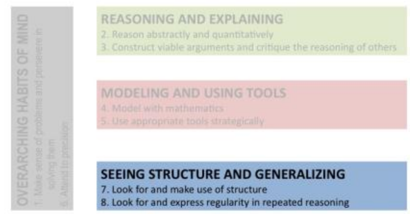
Kindergarten through Grade Twelve Standards for Mathematical Practice

Unit 5:

Seeing Structure and Generalizing (MP7 and MP8)

CALIFORNIA DEPARTMENT OF EDUCATION
Tom Torlakson, State Superintendent of Public Instruction

CCSS Mathematical Practices



Unit 5 Learning Objectives

- You will be able to describe why, to be successful in mathematics, all students need to see structure and generalize.
- You will be able to explain what it means for students to look for and make use of structure.
- You will be able to explain what it means for students to look for and express regularity in repeated reasoning.

Unit 5 Overview

- Unpacking MP7 and MP8
- Structure, Repeated Reasoning, and Generalization
- Making Sense of a Growing Pattern
- Geometry Examples of Structure and Generalization
- Performance Tasks and Student Work
- Summary and Reflection

5.0 Unpacking MP7 and MP8

Read MP7 and MP8

- Highlight key words or phrases that seem particularly cogent to you or that puzzle or intrigue you
- Make a note of questions you have about particular parts of these two mathematical practices.
- Consider in particular how the two practices are related.

Small Group Discussion

- What key words or phrases did you highlight? Why were these important to you?
- What questions do you have about these two mathematical practices?
- How are the two standards related?
- Which strategies from MP7 and MP8 do your students currently use?
- What challenges do you anticipate in your efforts to support students in meeting the demands of these two practices?

5.1 Seeing Structure and Using Repeated Reasoning and Generalization

Consecutive Sums

- Think about the following problem individually for 3 minutes.
- Then work on the problem in your table group for 15 minutes.

Some numbers can be written as a sum of consecutive positive integers:

$6 = 1 + 2 + 3$

$15 = 4 + 5 + 6$
$\quad = 1 + 2 + 3 + 4 + 5$

Which numbers have this property? Explain.

Consecutive Sums at Your Grade Span

Refer to "Consecutive Sums" **(Handout 5.1.1).**

- Group by grade span (K–2; 3–5; 6–8; 9–12).
- Consider how this problem might be presented for your grade span.
- Rewrite the problem and question for your grade span. What might you expect your students to do on this problem?
- How are students using repeated reasoning, structure and generalization in working on this problem?

Modifications for Consecutive Sums

Consider the following modifications for English learners, underperforming students, and those with special needs:

- Discuss what consecutive means by giving an example and non-example.
- Discuss other terms students might not understand, such as "conjecture" and "look for patterns."
- Provide base 10 blocks for students to make the sums manipulatively.
- Ensure any modification does not reduce the cognitive demand of the task.

Mathematical Structure in Number Systems

The Whole Numbers (0, 1, 2, 3…) satisfy:

Closure, the **Commutative Property**, and the **Associative Property** for addition and multiplication

0 is the **Additive Identity**, 1 is the **Multiplicative Identity**, and the **Distributive Property** connects multiplication and addition.

When we extend to integers we also have the **Additive Inverse Property**.

When we extend to the Rational Numbers, we also have the **Multiplicative Inverse Property**.

Generalization

"Generalizations are the lifeblood of mathematics."
Mason, et al., 2011

Although we regularly make generalizations in real life, they are especially essential in mathematics. By examining examples such as:

$a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^5$
$a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7$ and so on

One can conclude that:

am × an = am+n

…thus generalizing to all cases for a specific domain for the base "a" and the exponents "m" and "n."

Generalization

In mathematics, generalization can be both a process and a product.

- When one looks at specific instances, notices a pattern, and uses inductive reasoning to conjecture a statement about all such patterns, one is ***generalizing***.
- The symbolic, verbal, or visual representation of the pattern in your conjecture might be called a ***generalization***.

Generalization

"Generalizing is the process of "seeing through the particular" by not dwelling in the particularities but rather stressing relationships …. whenever we stress some features we consequently ignore others, and this is how generalizing comes about. "

Mason, et al., 2011

Generalization

When a student notices that the sum of an even and an odd integer always results in an odd integer, that student is generalizing.

Generalizations such as this allow students to think about computations independently of the particular numbers that are used. Without this, and many other generalizations we make in mathematics from the early grades, all of our work in mathematics would be cumbersome and inefficient.

Reflection

In your Metacognitive Journal, reflect on how the structure and generalization mathematical practices are inextricably linked.

Generalization in a Second Grader (Optional Activity)

Watch the video included in "Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School" (Carpenter, et al., 2003), and consider the following questions:

- In the example ½ + 11 - 11, does Susie work from left to right in her calculations?
- What can you infer about her understanding of the structure of the number system?
- What property is she using?
- What does she seem to understand about the use of a variable as an indicated element of an infinite set?
- How does she use generalization in her work?

5.2 Making Sense of a Growing Pattern

- Refer to Square Tiles (**Handout 5.2.1**)
- Solve the problem in your table groups.
- Present solution(s) to whole group.

Sixth Grade Student

View a video of sixth grader Tamara solving the same problem:

- Pre-interview Task: Find the 10^{th} and 100^{th} terms in the pattern.
- Post-interview Task: Solve all parts of the problem.

The pre-interview was in September and the post-interview was in May of Tamara's sixth grade year.

Pre- and Post-Interview Videos

As you watch, consider Tamara's use of the following:

- Strategies for finding a generalization
- Visualization and structure of the pattern
- Repeated reasoning
- Facility in using symbolic representations
- Structuring of arithmetic computations to track work

Both videos available on the Brokers of Expertise Web site
http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11862

Post-Video Discussion

In groups, discuss the following:

- What strategies for finding a generalization does Tamara use in the post- interview? How do these compare to the pre-interview?
- What is the role of visualization in her work? Of structure of the pattern?
- How is repeated reasoning used to get a generalization?
- What is Tamara's facility in using symbolic representations for the square tiles pattern?
- How does Tamara structure her arithmetic computations to keep track of her work?

Summary and Reflection

In grade span groups, reflect on the critical thinking and problem solving skills used to solve the problem.

- What critical thinking and problem solving skills did Tamara use?
- How are these different from the ones you used to solve the same problem?
- How might you plan instruction so that your students make sense of growing patterns?

5.3 Structure and Generalization in Geometry

Structure in geometry involves understanding basic properties of geometric figures and how they relate to each other. It also entails understanding geometric relationships such as perpendicularity and parallelism, and the connection between angle relationships.

Generalization in geometry is exemplified in a well-known middle school activity where students tear off the three corners of a triangle and arrange them to form a straight line. This is just one example of two mathematical practices at work in the geometric context.

5.3 Structure and Generalization in Geometry

Refer to the Geometry examples for your grade level span **(Handouts 5.3.1 or 5.3.2)**

- In grade span groups, explore the problem presented.
- Share problems as a whole group and discuss the use of structure or generalization in finding solutions.

5.3 Summary and Reflection

In your Metacognitive Journal or in grade span groups, respond to the following questions :

- How is structure related to the example for your specific grade span?
- What is the geometric structure that is the focus in the problem?
- How might a student use repeated reasoning?

5.4 Performance Tasks and Student Work

Refer to the MARS performance tasks and corresponding student work **(Handout 5.4.1–5.4.4)**.

- These tasks require students to determine patterns in growing figures and generalize from how they count objects to form a general rule for a pattern.
- Note that each task focuses on patterning problems in which one has to determine a relationship for the n^{th} term of a sequence.

Performance Tasks and Student Work

- In grade spans, first SOLVE the problem, then examine the student work.
- As a group, respond to questions.
- Be prepared to share sample student work and your analysis with the whole group.

Making Connections

By examining students' efforts to see structure and generalize, you also examined their ability to communicate their thinking and construct viable arguments to support their claims.

This experience connects directly to the writing standards in the CCSS for English Language Arts.

Differentiating Instruction

The following strategies support students, especially English learners and students with special needs, in communicating their thinking effectively.

- Allow students to write responses in their native language.
- Allow student responses to a task be a draft only. In class, examine some student work that needs revision and some that is acceptable or exemplary. Have students discuss these sample student works in small groups and then debrief with the whole class, focusing on how to improve explanations that need revision. Then allow students to revise their first draft on the task individually.

Differentiating Instruction

– Construct questions about student explanations that help the student focus on what is missing or is not clear. Use sticky notes so that the student can revise the work.

– Possible scaffolding questions to help students develop and articulate structure for a given pattern include:
  - How might you extend this pattern? Why extend it that way?
  - What stays the same and what changes in your pattern?
  - Is there another way to extend the pattern? How so?

5.5 Summary

To summarize what you have learned about MP7 and MP8, view the following collection of videos in which use of structure and generalizing from repeated reasoning are exemplified.

- Note any instances in which the students or the teacher demonstrate the use of **structure** or **repeated reasoning** in their work.

Elementary Example

After viewing the video, respond to the questions below:

http://insidemathematics.org/index.php/classroom-video-visits/public-lesson-number-operations/178-multiplication-a-divison-problem-3-part-a?

- How are students using mathematical structure in this video? How does the teacher reinforce it?
- What is the role of figures in the number talk? How does it reinforce structure?
- What is the double-double strategy?

Middle School Example

After viewing the video, respond to the questions below:

http://insidemathematics.org/index.php/classroom-video-visits/public-lessons-numerical-patterning/219-numerical-patterning-introduction-part-a

- How does Griffin work backwards to find the x value for a y value of 0?
- What is the mathematical structure issue related to the student discussion of x3 – 3 vs. 3x – 3 for the rule? How do students explain their thinking?

High School Example

After viewing the video, respond to the questions below:

http://insidemathematics.org/index.php/classroom-video-visits/public-lessons-properties-of-quadrilaterals/297-properties-of-quadrilaterals-tueasday-introduction-part-a

- This investigation is quite challenging. The students must determine what the diagonals would be to create each possible quadrilateral. Investigate this problem for yourself. Record in your metacognitive journal what you discover, and respond to the following questions:
- How is the structure of each individual quadrilateral determined by the diagonals?
- What kind of figure is determined if the two diagonals are of equal length? What kind of figure is determined if the diagonals are perpendicular to each other? How are diagonals of a trapezoid related to each other?
- How do students make sense of these relationships?

5.5 Summary

- In Unit 5 you have considered MP7 and MP8, the practices concerning structure and repeated reasoning and generalization.
- **Structure** refers to students' understanding and using properties of number systems, geometric features and relationships, and patterns of a variety of types, to solve problems.
- **Generalization** refers to the process of noticing repeated patterns or attributes, and using those to abstract and express general methods, expressions or equations, or relationships.

5.5 Reflection

In your metacognitive journal or in grade span groups, respond to these questions:

- Discuss your understanding of the two structure and generalization practices and how they work together .
- How will you begin to support your students to be successful in using structure and generalizing?
- What type of support will you need to make this happen?

California's Common Core State Standards for Mathematics