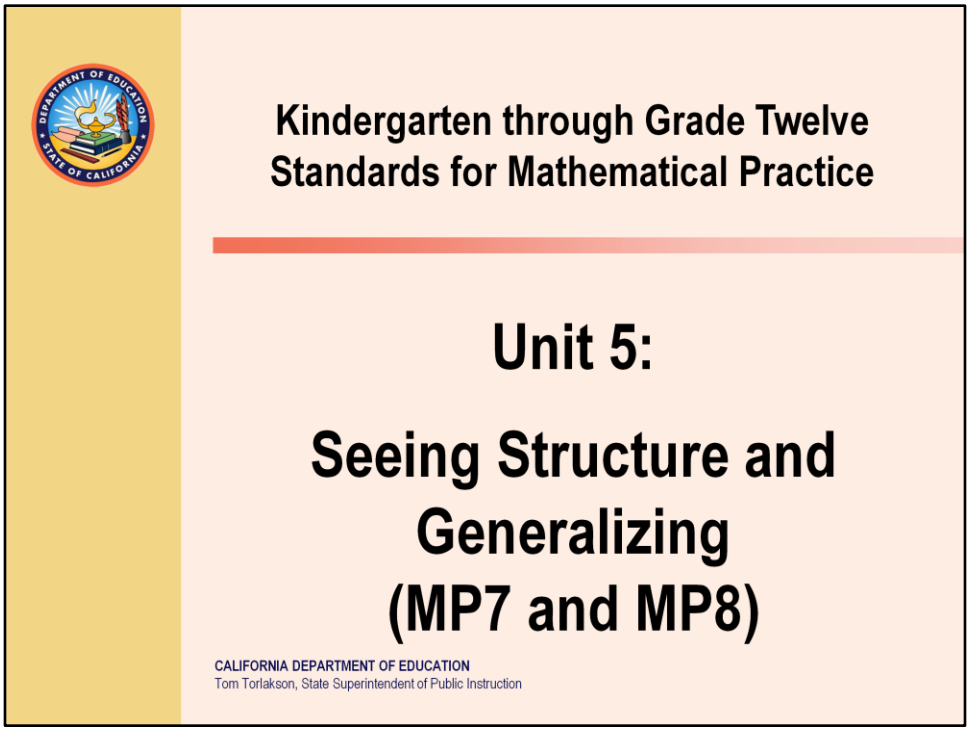
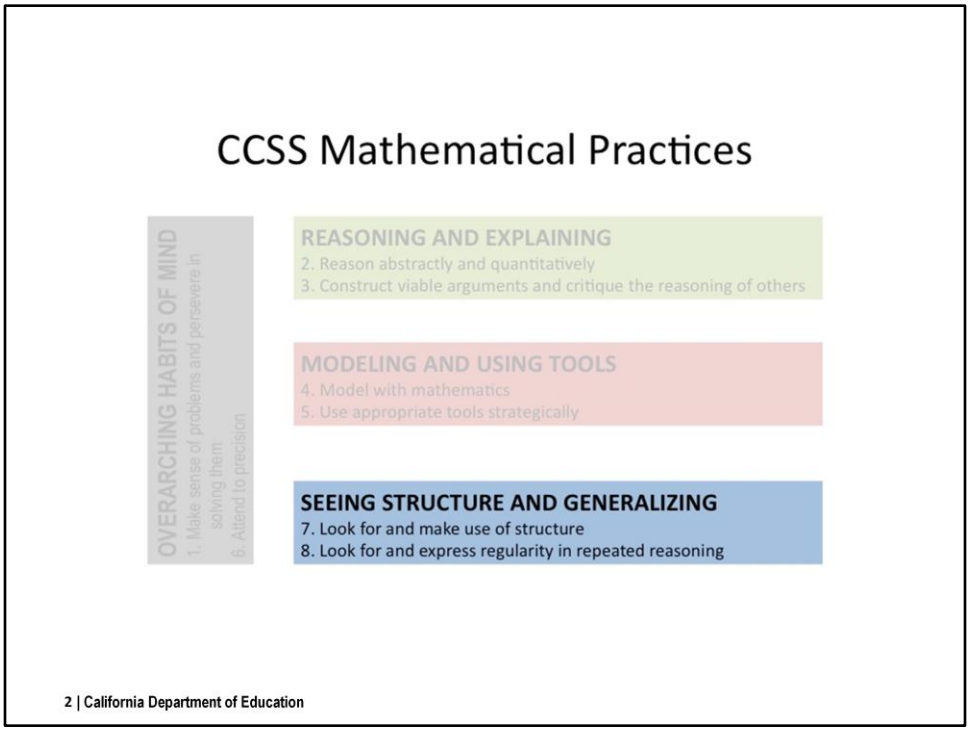


Talking Points:

Welcome to Unit 5.



Talking Points:

- In this unit, we will review the "Seeing Structure and Generalizing" practices; MP7 and MP8.

Facilitator Note:

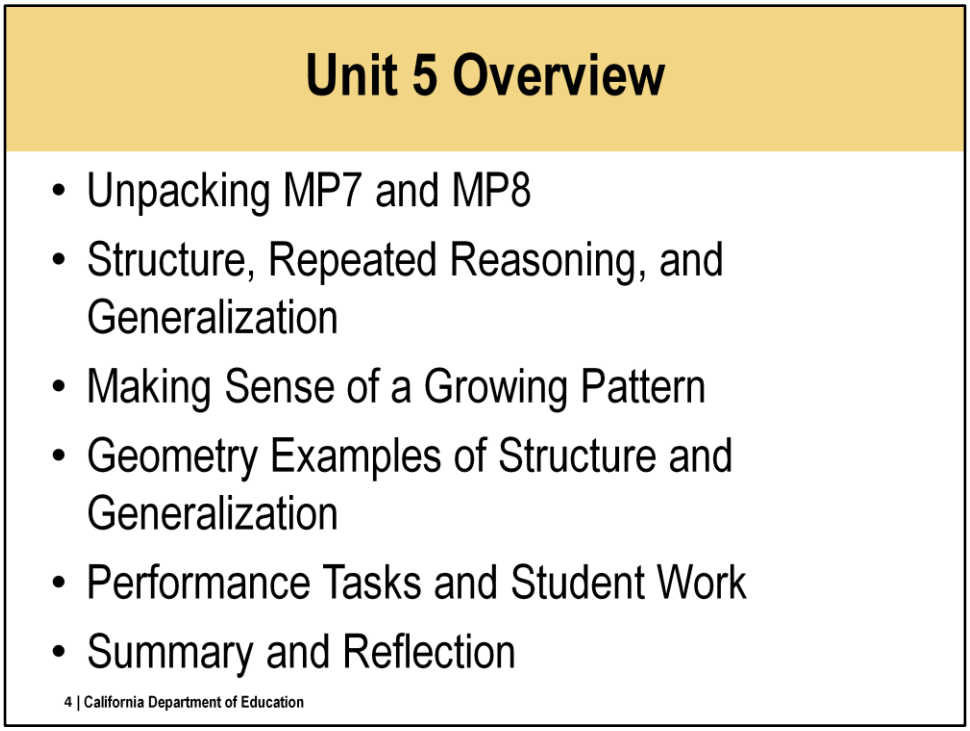
- Refer participants to the "CCSS Mathematical Practices" handout used in the previous section (**Handout 2.0**).

Unit 5 Learning Objectives

- You will be able to describe why, to be successful in mathematics, all students need to see structure and generalize.
- You will be able to explain what it means for students to look for and make use of structure.
- You will be able to explain what it means for students to look for and express regularity in repeated reasoning.

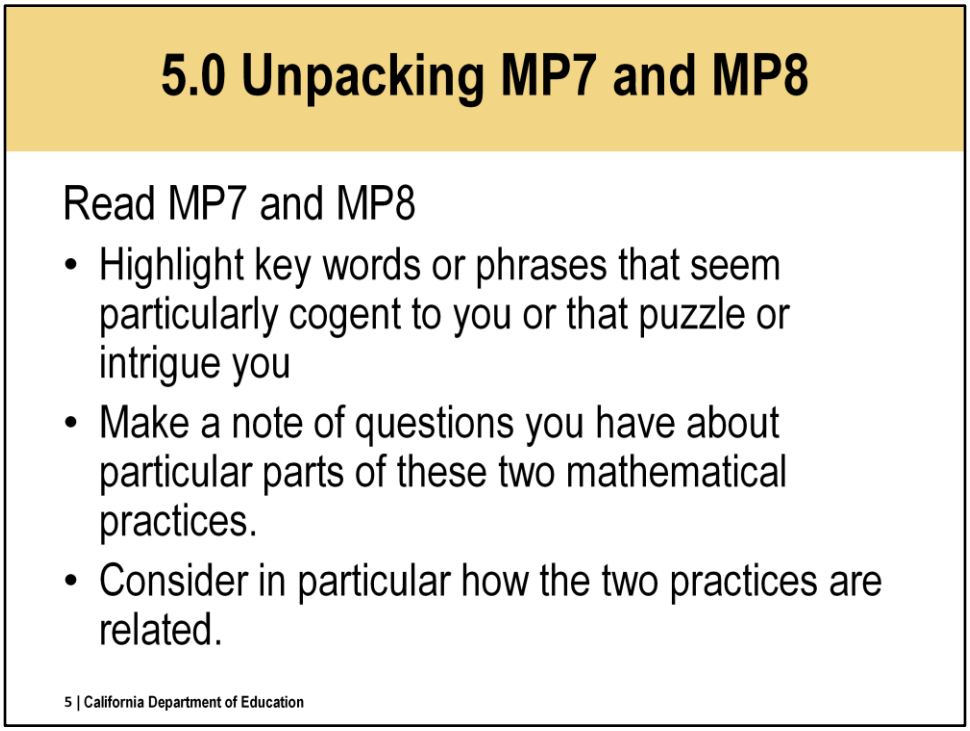
Talking Points:

- There are 3 learning objectives for this unit that focus on looking for and making use of mathematical structure and looking for and expressing regularity in repeated reasoning so that one can generalize a process or result.
- By the end of this unit…[review bullets on slide]



Talking Points:

- This unit contains the following sections.
  - First, we will unpack the contents of each MP individually.
  - Second, we will look at an example of looking for mathematical structure while using repeated reasoning and generalization in an open-ended problem adapted for your grade span. Generalization will also be discussed in this section.
  - Next we will examine a typical growing pattern that is linear in nature. After solving the problem yourselves, you will watch and analyze two video clips of a 6th grader solving the same problem before and after instruction.
  - The fourth section looks at examples of both structure and generalization in the context of geometry so that you have both number and geometric contexts in which to experience both practices.
  - The next section provides open-ended performance tasks for your grade span and student work on those tasks for you to evaluate.
  - The final section provides a summary and a video relevant to your grade span for you to analyze for evidence of MP7 and MP8.

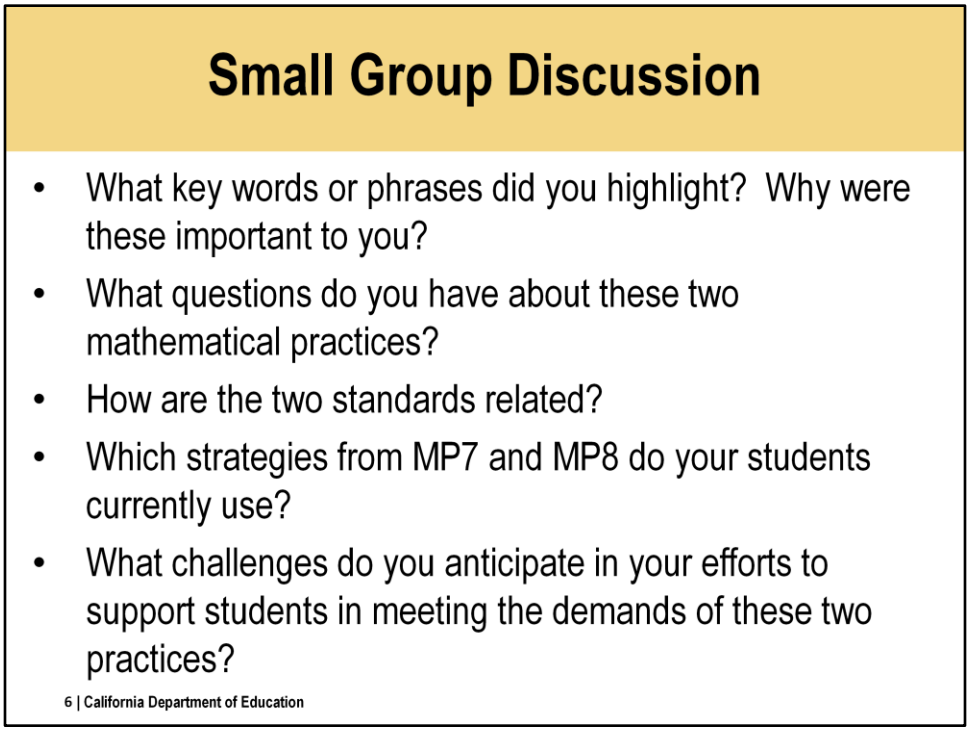


Facilitator Notes:

- Allow approximately 15 minutes to complete this activity.

Talking Points:

- Take a few minutes to review the modeling and using tools practices: "Unpacking MP7 and MP8" (**Handout 5.0.1**).
- As you read, note questions about particular parts of these two mathematical practices and highlight key words or phrases that puzzle, intrigue, or seem particularly cogent to you.
- Consider also how these two standards are related.



Facilitator Notes:

- Present and discuss these questions to the group.
- Consider having a whole group discussion, a table discussion, or pairs discussion followed by a whole group sharing.
- If participants are keeping a metacognitive journal, have them note responses to bullets 2–5 in the journal.

5.1 Seeing Structure and Using Repeated Reasoning and Generalization

Consecutive Sums

- Think about the following problem individually for 3 minutes.
- Then work on the problem in your table group for 15 minutes.

Some numbers can be written as a sum of consecutive positive integers:

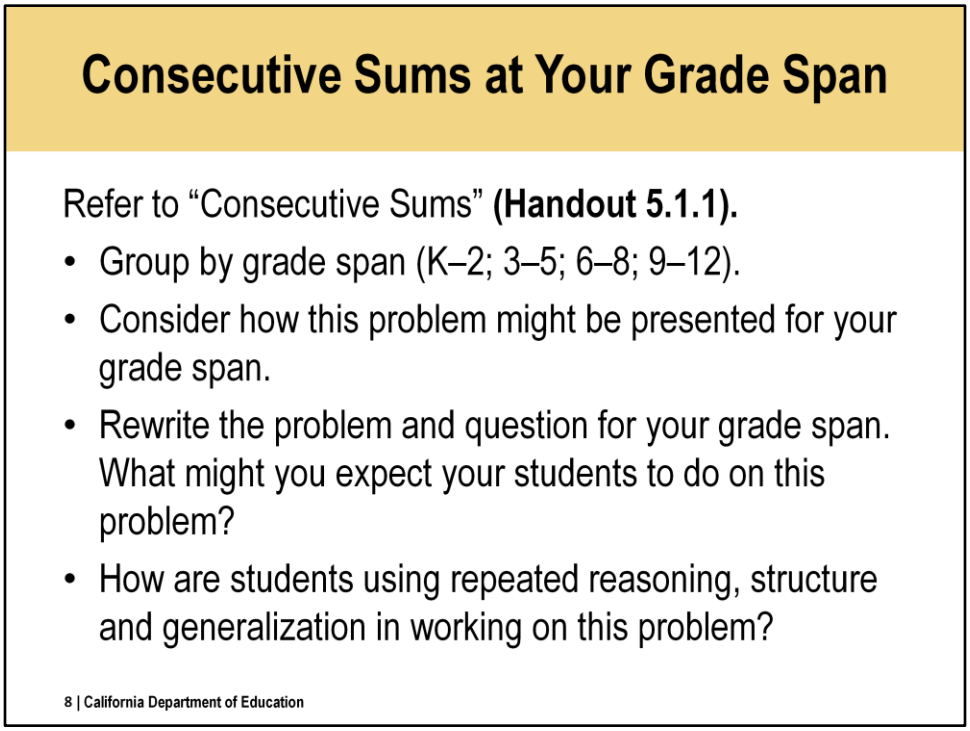
$6 = 1 + 2 + 3$

$15 = 4 + 5 + 6$
$= 1 + 2 + 3 + 4 + 5$

Which numbers have this property? Explain.

Facilitator Notes:

- Present the problem on the slide and have table groups work for about 15–20 minutes on the problem.
- If time permits, have participants make status posters of what they have found.
  - In that case, do a gallery walk and then have a discussion.
  - Or, select posters to discuss in order from least sophisticated (mathematically) to more sophisticated, and have teachers explain their thinking.



Facilitator Notes:

- Group participants by grade span and refer participants to "Consecutive Sums" **(Handout 5.1.1).**
- Have participants rewrite the problem and question(s) for their grade span.
- Share sample rewritten problems with the whole group by increasing grade spans.
- Discuss how students at each grade span are engaged in use of structure, repeated reasoning and generalization on working on the problem.
- Consider having participants respond to the last question in their Metacognitive Journals.

Modifications for Consecutive Sums

Consider the following modifications for English learners, underperforming students, and those with special needs:

- Discuss what consecutive means by giving an example and non-example.
- Discuss other terms students might not understand, such as “conjecture” and “look for patterns.”
- Provide base 10 blocks for students to make the sums manipulatively.
- Ensure any modification does not reduce the cognitive demand of the task.

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Facilitator Notes:

- Discuss possible modifications.
- Elicit feedback from participants.

Mathematical Structure in Number Systems

The Whole Numbers (0, 1, 2, 3…) satisfy:

Closure, the **Commutative Property**, and the **Associative Property** for addition and multiplication

0 is the **Additive Identity**, 1 is the **Multiplicative Identity**, and the **Distributive Property** connects multiplication and addition.

When we extend to integers we also have the **Additive Inverse Property**.

When we extend to the Rational Numbers, we also have the **Multiplicative Inverse Property**.

Talking Points:

- The **structure** of our number systems refers to the way they work.
- This includes the **place value** system in which we denote numbers and how different places are related to each other.
- Equally important are the properties that our numbers satisfy.

[review slide]

Facilitator notes:

- Discuss the properties and have teachers give one or two examples of each.
- Discuss **why** we need the additive inverse property to extend to the Integers, and **why** we need the multiplicative inverse property to extend to the Rational Numbers.

Generalization

"Generalizations are the lifeblood of mathematics."

Mason, et al., 2011

Although we regularly make generalizations in real life, they are especially essential in mathematics. By examining examples such as:

$a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^5$

$a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7$ **and so on**

One can conclude that:

$am \times an = am+n$

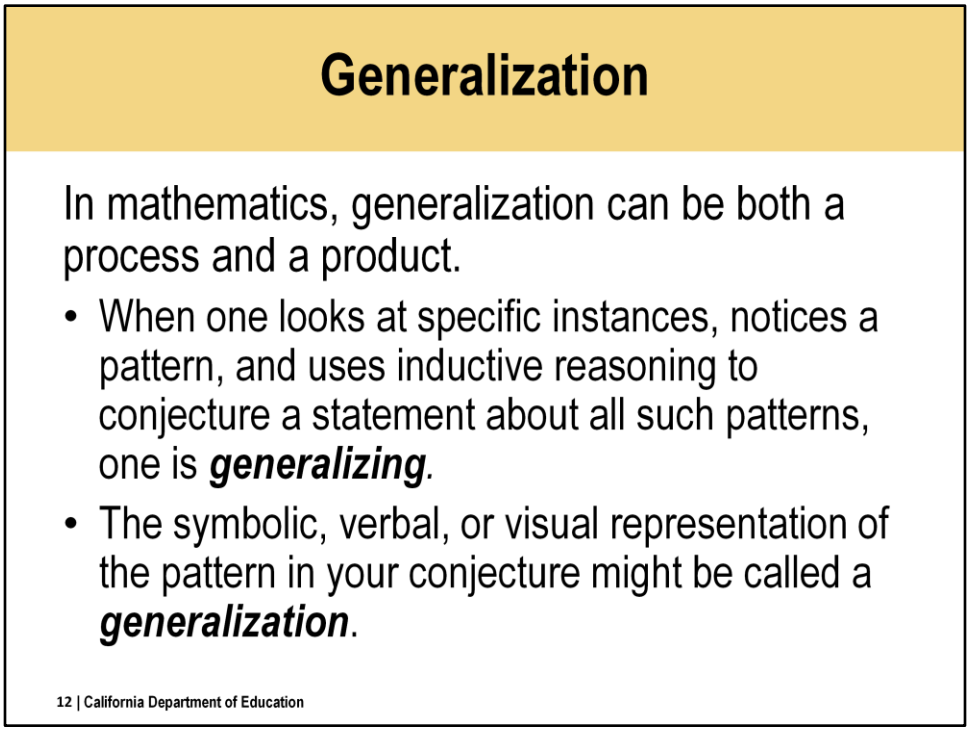
…thus generalizing to all cases for a specific domain for the base "a" and the exponents "m" and "n."

Talking Points:

- What do you think this quote means? [Discuss quote briefly as a whole group].

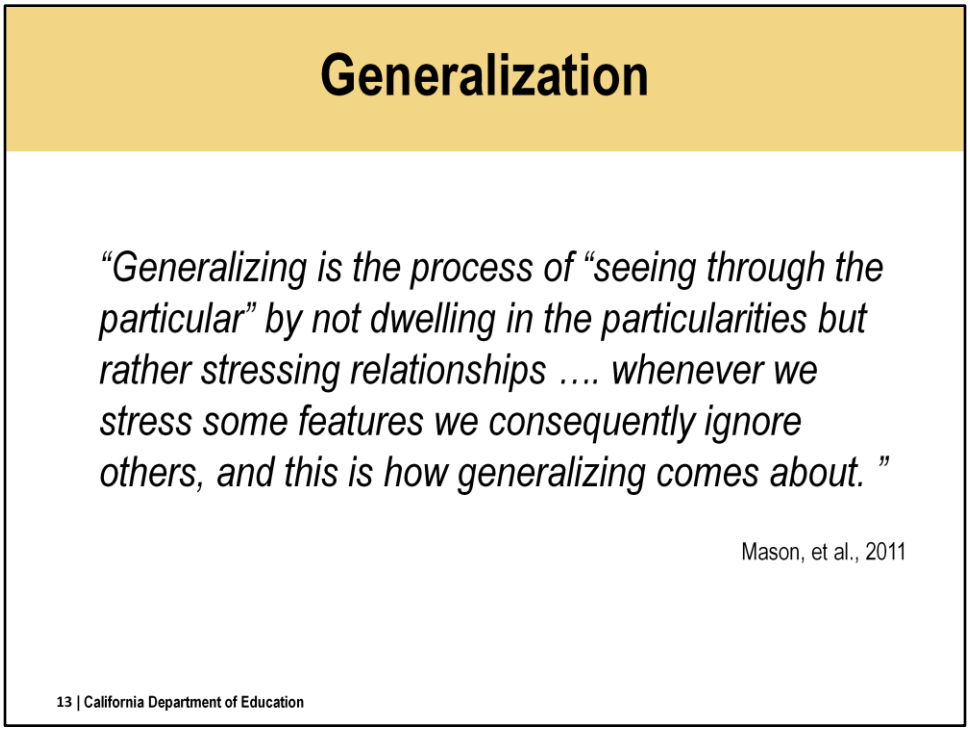
Facilitator Note:

- Review slide.
- Do a pair-share to discuss how generalization is used to derive laws of exponents such as in this example.



Facilitator Note:

- Review slide.



Facilitator Note:

- Ask participants to react to the quote on the slide.
- Consider a think-write-pair-share activity.

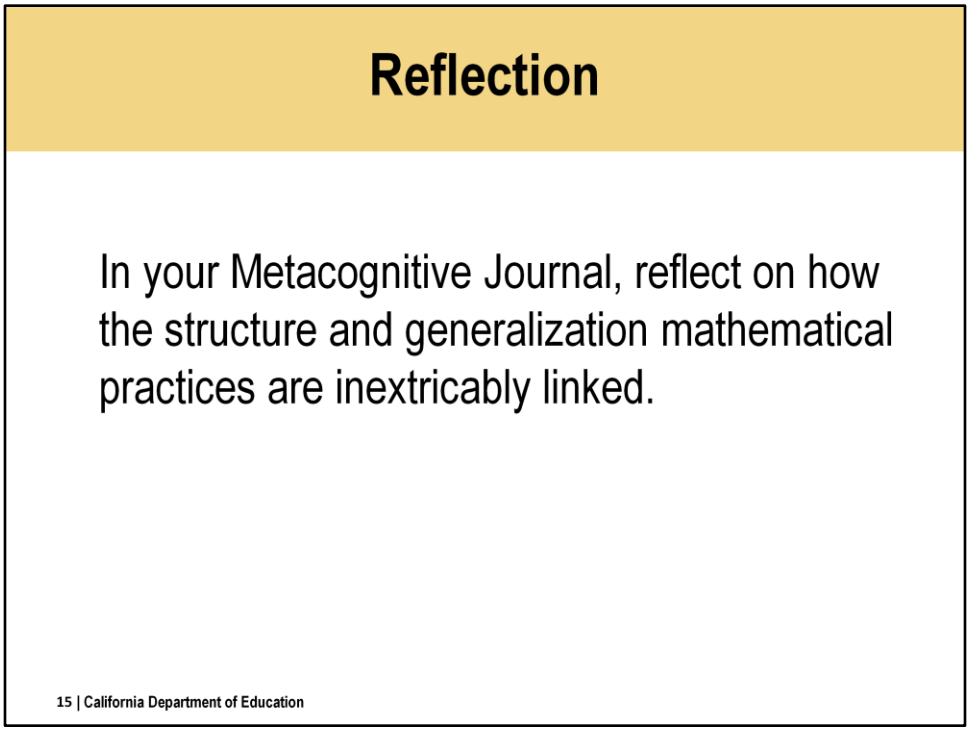
Generalization

When a student notices that the sum of an even and an odd integer always results in an odd integer, that student is generalizing.

Generalizations such as this allow students to think about computations independently of the particular numbers that are used. Without this, and many other generalizations we make in mathematics from the early grades, all of our work in mathematics would be cumbersome and inefficient.

Facilitator Notes:

- Review slide.
- Have participants think of another way generalization is used in the early grades or middle school.



Facilitator Notes:

- If time have a few comments read out.

Generalization in a Second Grader (Optional Activity)

Watch the video included in "Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School" (Carpenter, et al., 2003), and consider the following questions:

- In the example ½ + 11 - 11, does Susie work from left to right in her calculations?
- What can you infer about her understanding of the structure of the number system?
- What property is she using?
- What does she seem to understand about the use of a variable as an indicated element of an infinite set?
- How does she use generalization in her work?

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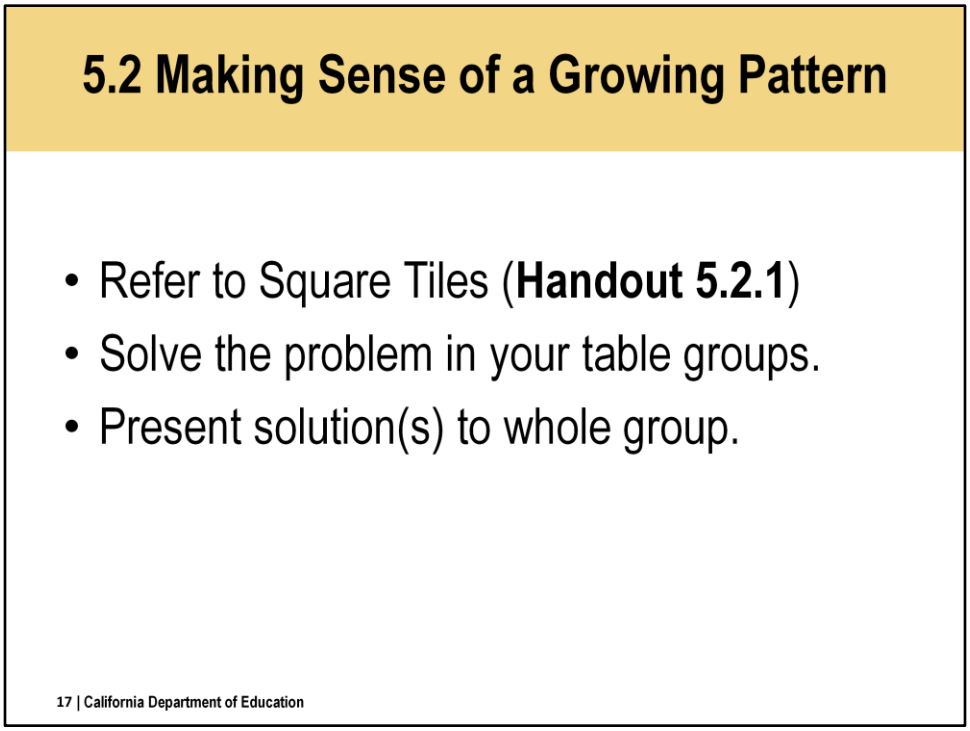
Facilitator Notes: This activity is optional (if time and if you can obtain the video):

- Refer to the publication, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School (Carpenter, et al., 2003) and on the included disc, view the video of Susie as she tries to justify a + b - b = a. Susie is a second grade student who has been learning in a Cognitively Guided Instruction environment.
- Discuss the questions on the slide after viewing the video.
- Consider assigning each question to a table to take the responsibility for beginning the discussion of that question, then all chime in.



Talking Points:

- Refer participants to Square Tiles (Handout 5.2.1)
- Give table groups time to solve the problem and generalize (approximately 15–20 minutes).
- Discuss the problem solution by having participants present solution(s), explaining how they counted the squares, to the whole group (using a document camera, if available).

Sixth Grade Student

View a video of sixth grader Tamara solving the same problem:

- Pre-interview Task: Find the 10^{th} and 100^{th} terms in the pattern.
- Post-interview Task: Solve all parts of the problem.

The pre-interview was in September and the post-interview was in May of Tamara's sixth grade year.

Talking Points:

- Now we will view the videos of a sixth grade student solving the same problem.
- In the pre-interview, the student had only to find the 10^{th} and 100^{th} term in the pattern.
- In the post-interview, she solved all parts of the problem.
- The pre-interview was in September and the post-interview was in May of her sixth grade year.

Pre- and Post-Interview Videos

As you watch, consider Tamara's use of the following:

- Strategies for finding a generalization
- Visualization and structure of the pattern
- Repeated reasoning
- Facility in using symbolic representations
- Structuring of arithmetic computations to track work

Both videos available on the Brokers of Expertise Web site
http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11862

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Facilitator Notes:

- Review slide prior to showing the two video clips.
- Consider assigning each table responsibility for one question.
- If time, pause between videos to allow for discussion.

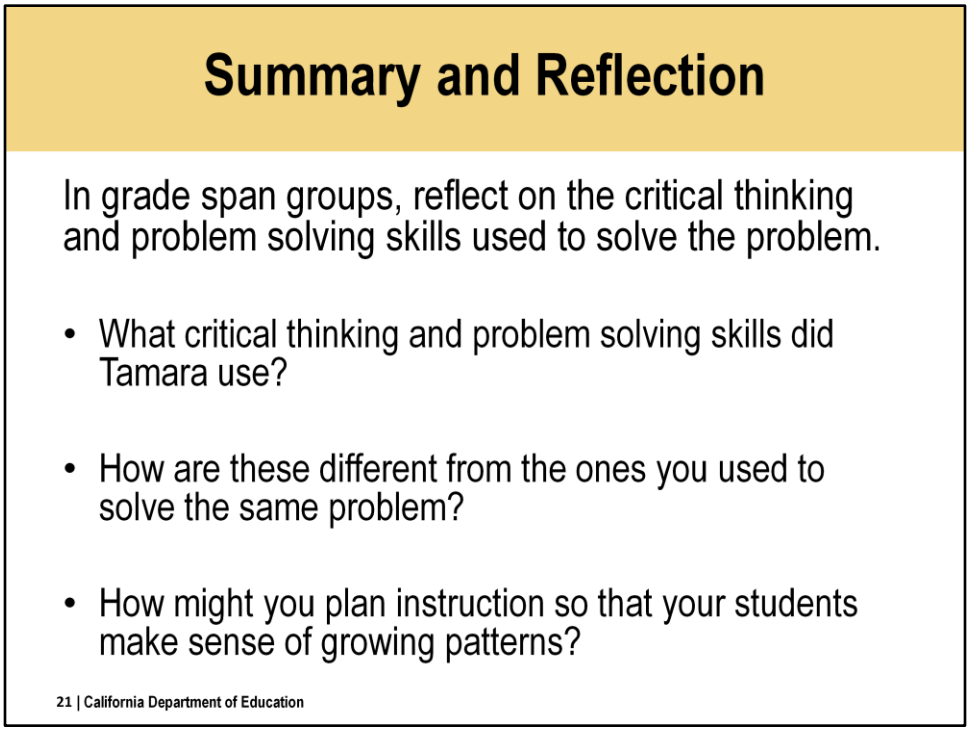
Post-Video Discussion

In groups, discuss the following:

- What strategies for finding a generalization does Tamara use in the post- interview? How do these compare to the pre-interview?
- What is the role of visualization in her work? Of structure of the pattern?
- How is repeated reasoning used to get a generalization?
- What is Tamara's facility in using symbolic representations for the square tiles pattern?
- How does Tamara structure her arithmetic computations to keep track of her work?

Facilitator Notes:

- Facilitate small group discussion.
- Compare and contrast the two videos.



Talking Points:

- In this section, you looked at various ways to obtain a general formula for a pattern, representing how students look at figures and see visual growth from different perspectives.
- For example, one student might see one square in the middle and the increase in the number of squares on each of the four spokes. Others might see an "X" figure and count the squares on each diagonal and subtract the one in the middle. They might even count the number of squares on one diagonal and notice one less square on the other diagonal.
- Knowing that students have different perspectives is important for teachers to consider, as students need time to explore and discover how to make sense of the growing pattern.

Facilitator Note:

- Have participants discuss the reflection questions.
- Have each group report key ideas.
- If you have multiple days for the workshop, give a homework assignment of one of the readings suggested below:

Rivera, F., "Changing the Face of Arithmetic: Teaching Children Algebra," *Teaching Children Mathematics. 12* no. 6 (2006): 306–311.

Rivera, F., Ferdinand D. & Becker, J., "Algebraic Reasoning through Patterns," *Mathematics Teaching in the Middle School. 15* no. 4 (2009): 212–221.

Rivera, F. & Becker, J., "Figural and Numerical Modes of Generalizing in Algebra," *Mathematics Teaching in the Middle School. 11* no. 4 (2005): 198–203.

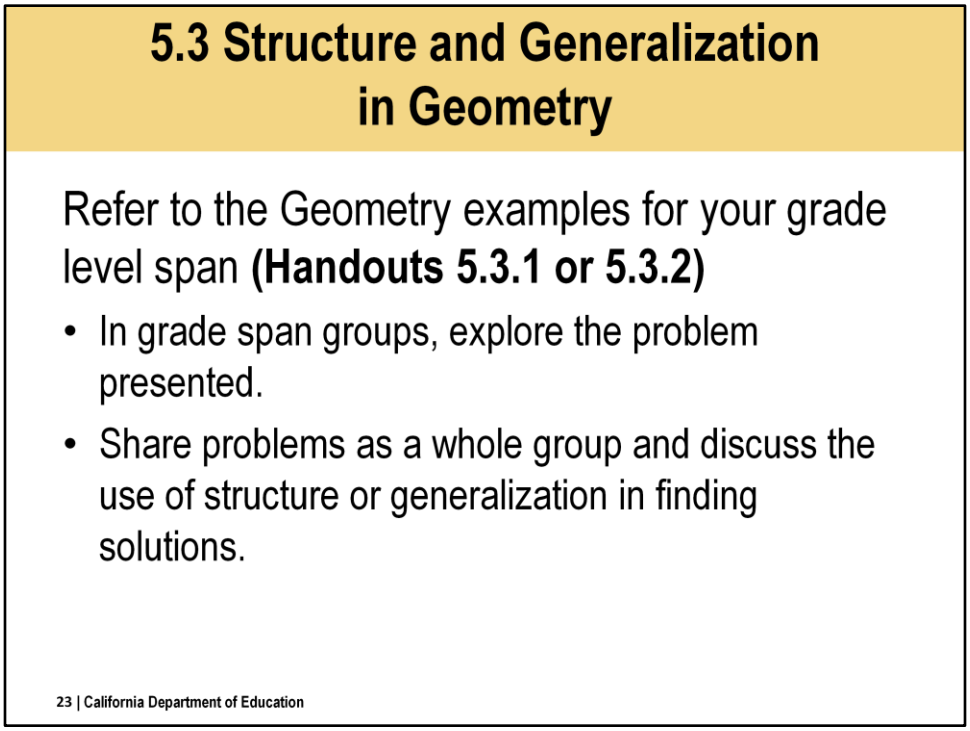
5.3 Structure and Generalization in Geometry

Structure in geometry involves understanding basic properties of geometric figures and how they relate to each other. It also entails understanding geometric relationships such as perpendicularity and parallelism, and the connection between angle relationships.

Generalization in geometry is exemplified in a well-known middle school activity where students tear off the three corners of a triangle and arrange them to form a straight line. This is just one example of two mathematical practices at work in the geometric context.

Facilitator Notes:

- Review slide.

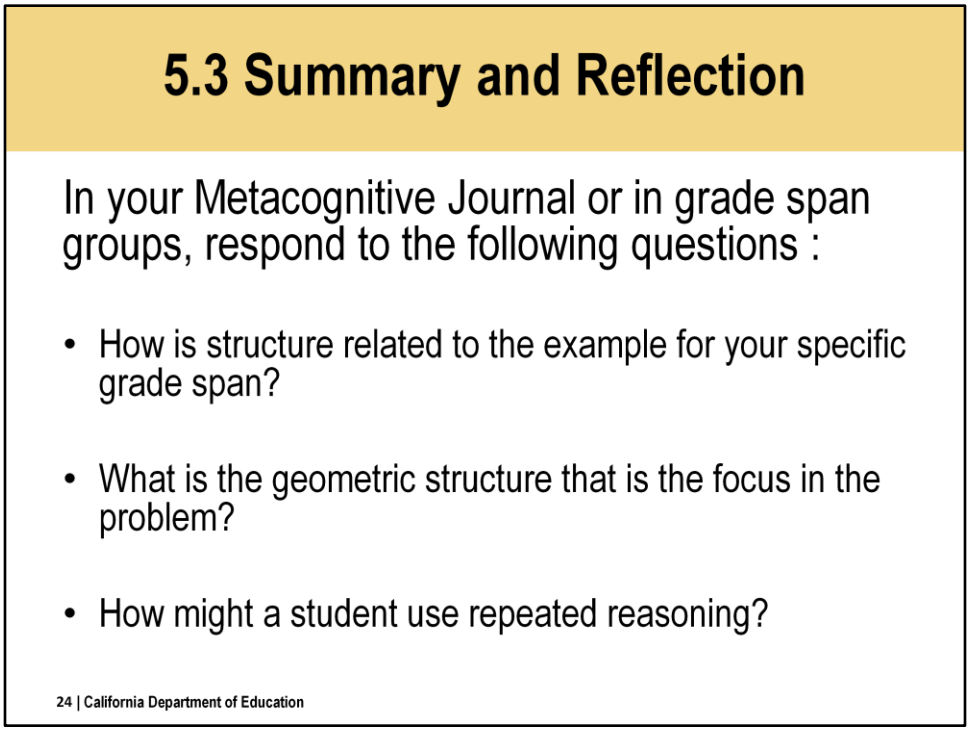


Facilitator Notes:

- Separate participants into elementary and secondary groups (separate rooms are ideal, but not required)
- Pattern blocks are needed for the K–2 example and graph paper and rulers are needed for the 3–5 example.
- Video capability is required for the 6–8 and 9–12 segments.

Talking Points:

- Refer to the Elementary Geometry Examples **(Handout 5.3.1)** and Secondary Geometry Examples **(Handout 5.3.2)** in your participant packet.
- Work on each problem and designate one person to present your problem and the analysis of where structure/generalization occur when we reconvene as a whole group.
- As time permits, examine the Geometry problems for other grade spans to identify a progression of geometric thought across the grades.



Facilitator Notes:

- If using the Metacognitive Journal, give some silent time to respond to these questions.
- If time, consider having volunteers read their entries aloud.
- Consider dividing into grade span groups for reflection and facilitate a general reporting out at the end.

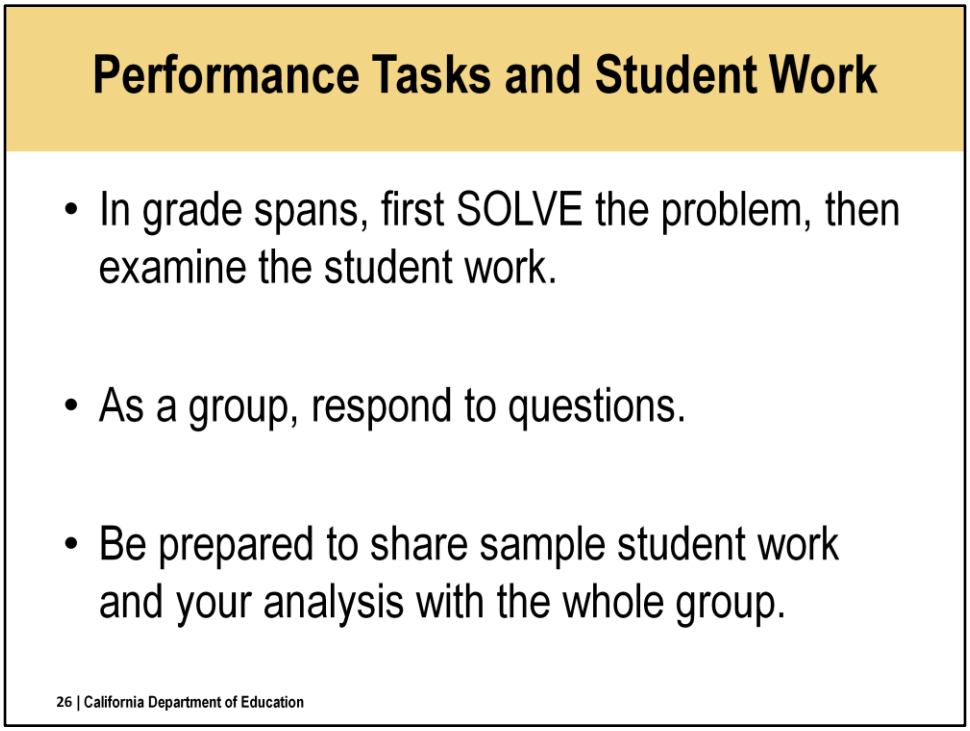
5.4 Performance Tasks and Student Work

Refer to the MARS performance tasks and corresponding student work **(Handout 5.4.1–5.4.4)**.

- These tasks require students to determine patterns in growing figures and generalize from how they count objects to form a general rule for a pattern.
- Note that each task focuses on patterning problems in which one has to determine a relationship for the n^{th} term of a sequence.

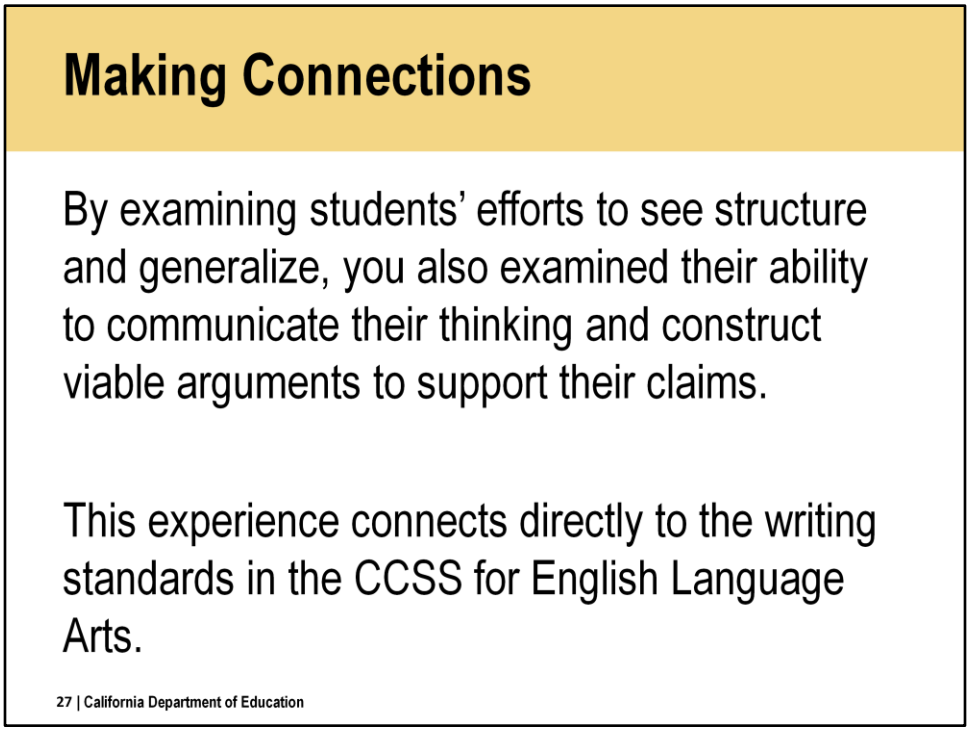
Talking Points:

- Let's take a look at some tasks that require students to determine patterns in growing figures and generalize from how they count objects to form a general rule for a pattern.
- Refer to the MARS performance tasks and corresponding student work **(Handouts 5.41–5.4.4)**.
- Although the tasks at each grade level are different, each task focuses upon patterning problems in which one has to determine a relationship for the n^{th} term of a sequence.



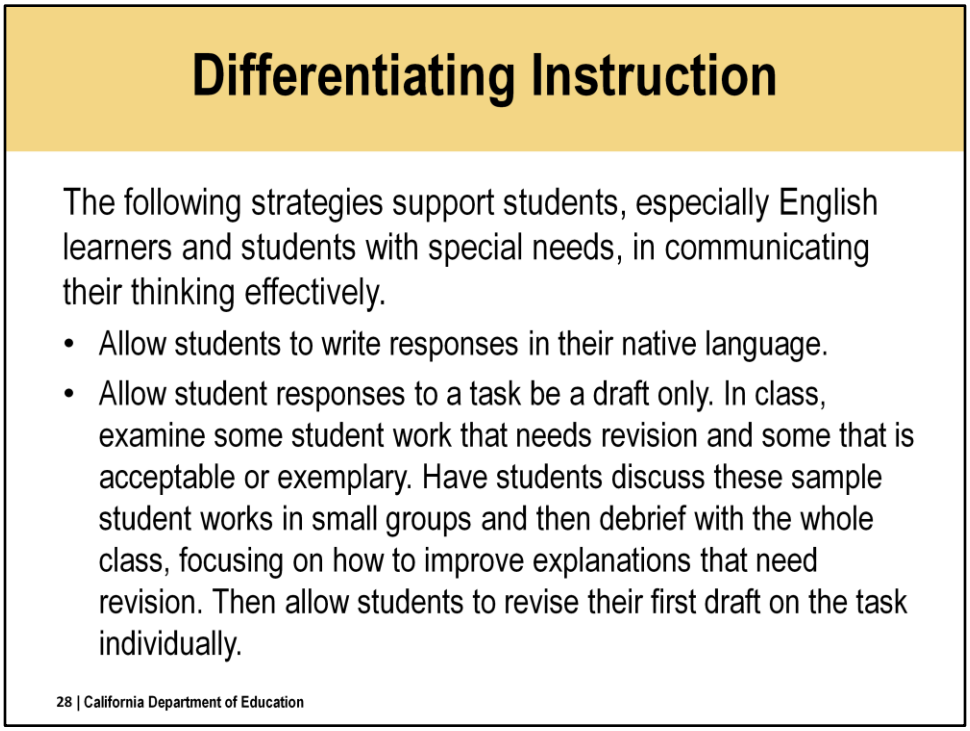
Facilitator Notes:

- Give the groups about 15–20 minutes to complete the task and analysis of student work and discuss answers to the questions.
- Reconvene as a whole group and have the grade span groups present their problems and selections of student work using a document camera (if available).



Facilitator Notes:

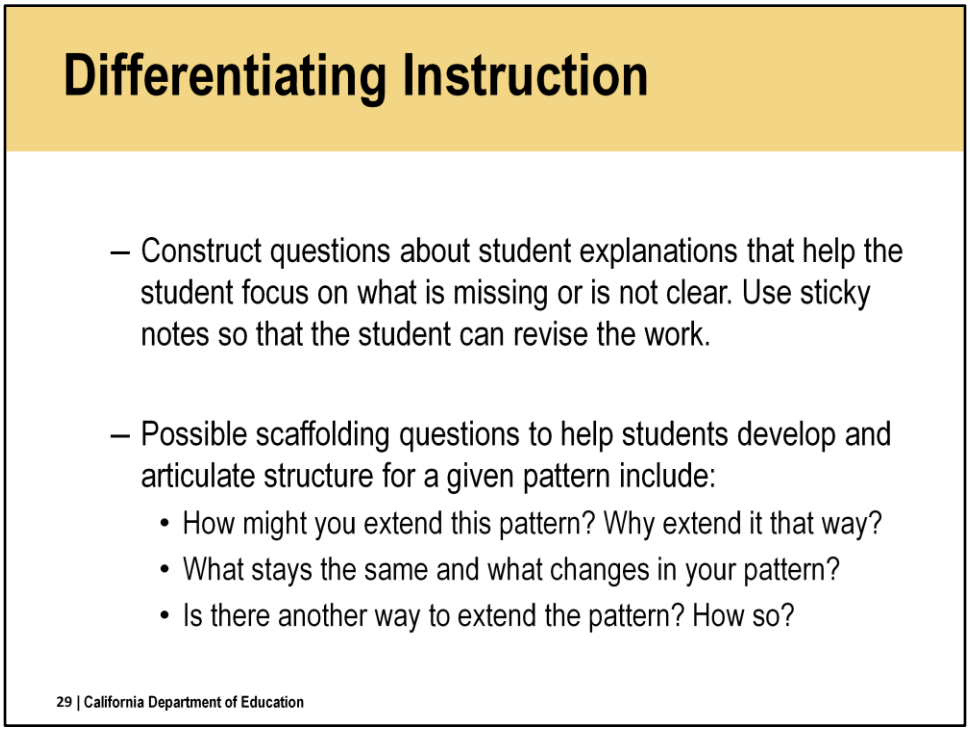
- Review this slide after the performance task activity is complete.



Talking Points:

- Consider how you would assist students in developing strategies for seeking visual and numerical structure and developing generalizations, whether verbal, symbolic, or numeric.
- The following strategies support students, especially English learners and students with special needs, in communicating their thinking effectively.
- Note that the strategies on the slide are only examples.

[review slide]



Facilitator Notes:

- Review slide.
- Have participants discuss the strategies in Slides 28 and 29 and have them add additional strategies that they have used effectively to differentiate instruction.

5.5 Summary

To summarize what you have learned about MP7 and MP8, view the following collection of videos in which use of structure and generalizing from repeated reasoning are exemplified.

- Note any instances in which the students or the teacher demonstrate the use of **structure** or **repeated reasoning** in their work.

Facilitator Notes:

- There is a video sample for each grade span: elementary, middle, and high. If you have a mixed group, you might choose one of these to view (perhaps the grade 6–8 one), or separate into different rooms if practical.
- After viewing the video, facilitate a short discussion about the questions relevant to the one you chose.

Elementary Example

After viewing the video, respond to the questions below:

http://insidemathematics.org/index.php/classroom-video-visits/public-lesson-number-operations/178-multiplication-a-divison-problem-3-part-a?

- How are students using mathematical structure in this video? How does the teacher reinforce it?
- What is the role of figures in the number talk? How does it reinforce structure?
- What is the double-double strategy?

Facilitator Notes:

- Show video and facilitate discussion of questions on the slide.

Middle School Example

After viewing the video, respond to the questions below:

http://insidemathematics.org/index.php/classroom-video-visits/public-lessons-numerical-patterning/219-numerical-patterning-introduction-part-a

- How does Griffin work backwards to find the x value for a y value of 0?
- What is the mathematical structure issue related to the student discussion of x3 – 3 vs. 3x – 3 for the rule? How do students explain their thinking?

Facilitator Notes:

- Show video **through Part C** and facilitate discussion of questions on the slide.

High School Example

After viewing the video, respond to the questions below:

http://insidemathematics.org/index.php/classroom-video-visits/public-lessons-properties-of-quadrilaterals/297-properties-of-quadrilaterals-tueasday-introduction-part-a

- This investigation is quite challenging. The students must determine what the diagonals would be to create each possible quadrilateral. Investigate this problem for yourself. Record in your metacognitive journal what you discover, and respond to the following questions:
- How is the structure of each individual quadrilateral determined by the diagonals?
- What kind of figure is determined if the two diagonals are of equal length? What kind of figure is determined if the diagonals are perpendicular to each other? How are diagonals of a trapezoid related to each other?
- How do students make sense of these relationships?

Facilitator Notes:

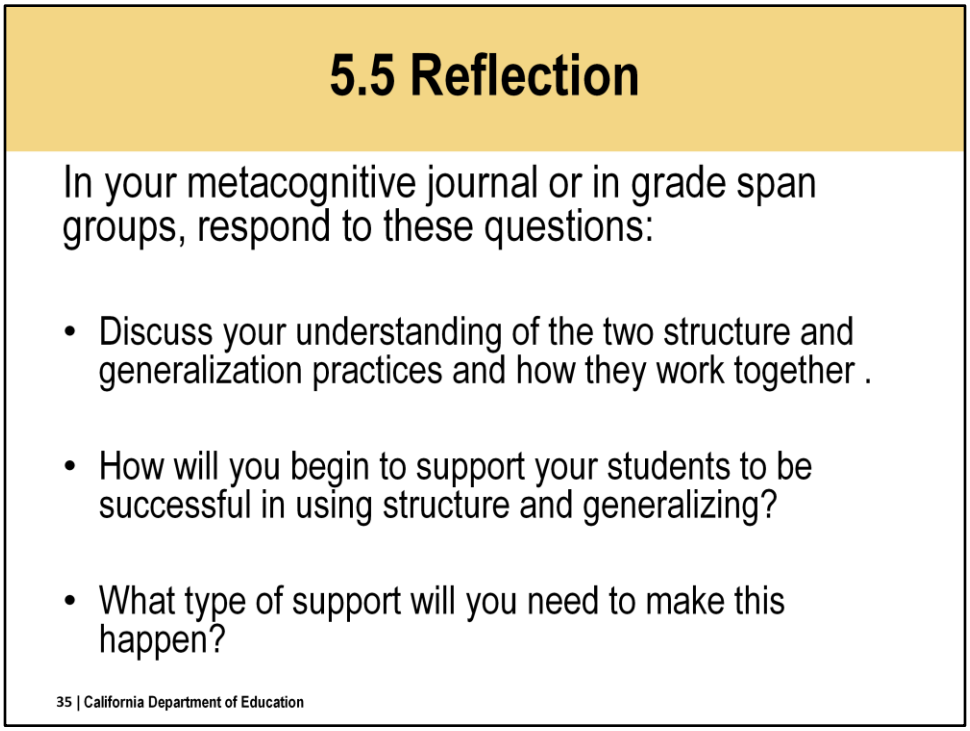
- Show video and facilitate discussion of questions on the slide.

5.5 Summary

- In Unit 5 you have considered MP7 and MP8, the practices concerning structure and repeated reasoning and generalization.
- **Structure** refers to students' understanding and using properties of number systems, geometric features and relationships, and patterns of a variety of types, to solve problems.
- **Generalization** refers to the process of noticing repeated patterns or attributes, and using those to abstract and express general methods, expressions or equations, or relationships.

Facilitator Notes:

- Review slide.



Facilitator Notes:

- Provide a brief summary of the whole unit.
- Provide quiet time for writing in metacognitive journal or a final evaluation, or divide into grade span groups, and then if time share out some teacher responses.

Talking Points:

- In Unit 5, you have considered MP7 and MP8, the practices concerning structure and repeated reasoning and generalization.
- Structure refers to students' understanding and using properties of number systems, geometric features and relationships, and patterns of a variety of types, to solve problems.
- Generalization refers to the process of noticing repeated patterns or attributes, and using those to abstract and express general methods, expressions or equations, or relationships.

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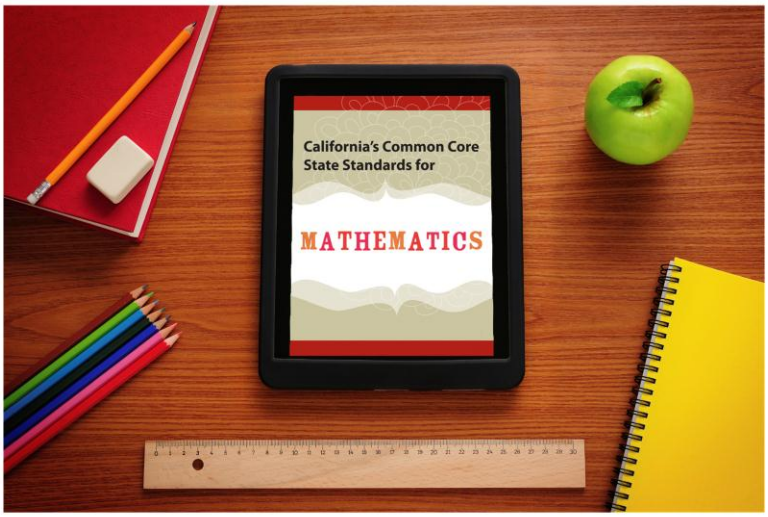

(TRANSITION SLIDE)