Pre-Assessment

1. The design of the Common Core SMP was informed by:
   b. The National Council of Teachers of Mathematics (NCTM) Process Standards
   c) Both "a" and "b"
   d) None of the above

2. When students decontextualize a situation they apply the following mathematical practice:
   a) Construct viable arguments and critique the reasoning of others
   b) Use appropriate tools strategically
   c) Reason abstractly and quantitatively
   d) Attend to precision

3. When students use clear definitions in a discussion, they apply the following mathematical practice:
   a) Look for and make use of structure
   b) Use appropriate tools strategically
   c) Reason abstractly and quantitatively
   d) Attend to precision

4. When students make assumptions and approximations to simplify a complicated situation they are applying the following mathematical practice:
   a) Model with mathematics
   b) Use appropriate tools strategically
   c) Reason abstractly and quantitatively
   d) Attend to precision
5. The SMP are:
   a) Primarily for gifted and talented students
   b) Primarily for high school students
   c) Important for all grade levels (K–12)
   d) Both “a” and “b”

6. In the CCSS, the SMP describe what it means to do and use mathematics.
   True
   False

7. Students are only responsible for mastering the content standards since it is not possible to assess the SMP.
   True
   False

8. The mathematical practice "Attend to precision" emphasizes using the correct number of decimal places for accuracy in student answers.
   True
   False

9. The mathematical practice "Modeling" is focused on showing students how to do a procedure.
   True
   False

10. Students are not expected to apply all eight SMP in every mathematical activity.
    True
    False
Principles and Standards for School Mathematics
Process Standards

Problem Solving

Instructional programs from pre-kindergarten through grade twelve should enable all students to:
• Build new mathematical knowledge through problem solving
• Solve problems that arise in mathematics and in other contexts
• Apply and adapt a variety of appropriate strategies to solve problems
• Monitor and reflect on the process of mathematical problem solving

Reasoning and Proof

Instructional programs from pre-kindergarten through grade twelve should enable all students to:
• Recognize reasoning and proof as fundamental aspects of mathematics
• Make and investigate mathematical conjectures
• Develop and evaluate mathematical arguments and proofs
• Select and use various types of reasoning and methods of proof

Communication

Instructional programs from pre-kindergarten through grade twelve should enable all students to:
• Organize and consolidate their mathematical thinking through communication
• Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
• Analyze and evaluate the mathematical thinking and strategies of others
• Use the language of mathematics to express mathematical ideas precisely

Connections

Instructional programs from pre-kindergarten through grade twelve should enable all students to:

• Recognize and use connections among mathematical ideas
• Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
• Recognize and apply mathematics in contexts outside of mathematics

Representation

Instructional programs from pre-kindergarten through grade twelve should enable all students to:

• Create and use representations to organize, record, and communicate mathematical ideas
• Select, apply, and translate among mathematical representations to solve problems
• Use representations to model and interpret physical, social, and mathematical phenomena

Handout 1.2.2

Intertwined Strands of Proficiency

Conceptual understanding: Comprehension of mathematical concepts, operations, and relations

Procedural fluency: Skill in carrying out procedures flexibly, accurately, efficiently, and approximately

Strategic competence: Ability to formulate, represent, and solve mathematical problems

Adaptive reasoning: Capacity for logical thought, reflection, explanation, and justification

Productive disposition: Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy

The Eight Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them**
   Making sense and persevering are habits of mind needed by all students to be successful learners of mathematics. Before a student can engage in mathematics, they need to make sense of what they are being asked to consider.

2. **Reason abstractly and quantitatively**
   Reasoning abstractly requires that students make sense of quantities and their relationships in problem situations. Students decontextualize and contextualize mathematics; they translate problem situations into symbols which they are able to manipulate and, as they manipulate the symbols, refer back to the problem situation to make sense of their work.

3. **Construct viable arguments and critique the reasoning of others**
   Constructing arguments requires that students use stated assumptions, definitions, and previous results. They make conjectures, justify their conclusions, and communicate them to others. They respond to the arguments of others.

4. **Model with mathematics**
   Modeling with mathematics requires that students make assumptions and approximations to simplify a situation, realizing these may need revision later, and that students interpret mathematical results in the context of the situation and reflect on whether they make sense.

5. **Use appropriate tools strategically**
   Using tools strategically requires that students are familiar with appropriate tools to decide when each tool is helpful, know both benefits and limitations, detect possible errors, and identify relevant external mathematical resources and use them to pose or solve problems.

6. **Attend to precision**
   Precision refers to the accuracy with which students use mathematical language and symbols as well as precision in measurement.

7. **Look for and make use of structure**
   Looking for structure refers to students’ understanding and using properties of number systems, geometric features and relationships, and patterns of a variety of types to solve problems.

8. **Look for and express regularity in repeated reasoning**
   Looking for regularity in repeated reasoning refers to the process of noticing repeated patterns or attributes and using those to abstract and express general methods, expressions or equations, or relationships.
Handout 1.2.4

Self-Reflection Survey

Name ______________________________________________________

**Common Core State Standards for Mathematical Practice**

How confident are you in supporting all students to be successful in the following eight mathematical practices:

<table>
<thead>
<tr>
<th></th>
<th>Not Confident</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Very Confident</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make sense of problems</td>
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<td>and persevere in solving</td>
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<td>Use appropriate tools</td>
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<td>strategically</td>
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<td>Look for and make use of</td>
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<td>regularity in repeated</td>
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<td>reasoning</td>
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</tbody>
</table>
Additional Unit 1 Resources

“K-12 California’s Common Core Content Standards for Mathematics”

“Mathematics, the Common Core, and Language: Recommendations for Mathematics Instruction for ELs Aligned with the Common Core” by Judit Moschkovich
http://ell.stanford.edu/sites/default/files/pdf/academic-papers/02-JMoschkovich%20Math%20FINAL.pdf

“Application to Students with Disabilities”

“The Special Edge: Summer 2012”

“Overview of the California English Language Development Standards and Proficiency Level Descriptors”
Available at http://www.cde.ca.gov/sp/el/er/documents/sbeoverviewpld.pdf
CCSS Mathematical Practices

OVERARCHING HABITS OF MIND
1. Make sense of problems and persevere in solving them
6. Attend to precision

REASONING AND EXPLAINING
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others

MODELING AND USING TOOLS
4. Model with mathematics
5. Use appropriate tools strategically

SEEING STRUCTURE AND GENERALIZING
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning
Handout 2.0

Compare Standards MP1 and MP6

Make sense of problems and persevere in solving them.

MP 1

Attend to precision.

MP 6
Standards for Mathematical Practice

MP 1 and MP 6

MP 1. Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP 6. Attend to precision
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they reach high school they have learned to examine claims and make explicit use of definitions.
Handout 2.1.1

Sierpinski’s Triangle

How many triangles do you see?

The diagram is called Sierpinski’s Triangle and is an example of a fractal. A fractal is a geometric pattern that is repeated in ever-smaller scale to produce irregular shapes and surfaces that cannot be represented by classical geometry.
**Handout 2.1.2**

**Teacher Reflections**

**Reflection prompt:** “What did you learn about teaching and learning algebraic thinking from your observations? Specifically, what happens when we give students free rein on a problem and is there value in that?”

**Teacher A:**
“I learned that if you guide students in a certain way, they are not given the opportunity to think on their own. They are used to just doing what the teacher is telling them to do. We saw how the guidance of a teacher makes a big difference in how students approach problems and what they use to solve them. It was great, because now I will be thinking closely about what questions I ask my students and how I am guiding them.”

**Teacher B:**
“I enjoyed seeing what can happen when we, in our ultimate wisdom, give students too much guidance. It was interesting to see what they can come up with on their own when we don't hand-hold. There are so many tools and great thoughts that our students have and we don't allow them to exercise these enough because we are so busy cramming new information down their throats without time for processing and working through it.”

**Teacher C:**
“I believe it is extremely valuable to give students the opportunity to work together. If students can figure out how to solve a problem on their own, or with a group, it is much more valuable to them than if I tell them how to do it. Students do not forget what they have figured out on their own.”

(Work cited from Pomona Unified School District math teachers 2011–12)
Handout 2.1.3

Posing Questions and Responding to Students

Sample Question in Primary

Teacher: Write a word sentence for the following picture:

Student Response: There are six in the picture.

Teacher 1: Right!

Teacher 2: Six what?

Teacher 3: Six what? Does anyone see something else? What else can you tell me about the picture?
Sample Question in Upper Elementary

**Teacher:** Students, draw a picture of one half of one third.

**Student Response:** Here is my picture:

![Diagram showing one third of one, one half of one third, and one half of one](image)

**Teacher 1:** Right, the shaded part at the end shows one half of one third!

**Teacher 2:** How do you know that your last picture shows one half of one third?

**Teacher 3:** Explain your thinking. Why do you have the shape drawn four times? Convince me that the last one illustrates one half of one third.
Handout 2.1.5

Posing Questions and Responding to Students

Sample Question in Middle School

**Teacher:** Students, solve the equation $2(x - 4) + 3 = x - 4$. Show all work.

**Student Response:** Here is my solution:

\[
\begin{align*}
2(x - 4) + 3 &= x - 4 \\
2x - 8 + 3 &= x - 4 \\
2x - 5 &= x - 4 \\
x &= 1
\end{align*}
\]

**Teacher 1:** Your solution is correct.

**Teacher 2:** How do you check to see if your answer is correct?

**Teacher 3:** Could you have done this problem another way? What could have been a different first step? Can you justify each step?
Handout 2.1.6

Posing Questions and Responding to Students

Sample Question in High School

**Teacher:** *Students, solve the inequality* \( x^2 - x - 6 < 0 \)

**Student Response:** *I decided to do this by graphing. I graphed the function* \( y = x^2 - x - 6 \) *below:*

![Graph](image)

…So, the answer is that the inequality \( x^2 - x - 6 < 0 \) occurs when \(-2 < x < 3\).

**Teacher 1:** *Where are your cases?*

**Teacher 2:** *This is correct. Why did you go with a graphing solution?*

**Teacher 3:** *If you graphed* \( y = |x^2 - x - 6| \), *what would that graph look like? How does that graph differ from the one you drew? How is the answer connected to the zeros of* \( y = x^2 - x - 6 \)?
Handout 2.2.1

The Hook to Persevere

Primary Task Set

Task 1: Measure the length of your desk in centimeters.

Task 2: Measure the length of your desk using your hand span. Measure the length of the desk again using another student's hand span. Are the measures the same? Why might they be different?
Upper Elementary Task Set

**Task 1:** On a coordinate grid, graph and label the following points:

\[(3, 4), (-3, 4), (4, -3), (-4, -3), (3, -4), (4, 3), (-3, -4), (-4, 3)\]

**Task 2:** Label the points drawn on the coordinate grid below. Connect the points to make a picture frame. Describe the line segments you created to make your frame.
Middle School Task Set

Task 1: Determine the volume of a rectangular prism that is 9 ft long, 12 feet wide and 15 inches deep. What would be the volume of a rectangular prism that was 1.5 times as long and 1.5 times as wide as the original prism?

Task 2: The 7th graders at Sunview Middle School were helping to renovate a playground for the kindergartners at a nearby elementary school. City regulations require that the sand underneath the swings be at least 15 inches deep. The sand under both swing sets was only 12 inches deep when they started. The rectangular area under the small swing set measures 9 feet by 12 feet and required 40 bags of sand to increase the depth by 3 inches. How many bags of sand will the students need to cover the rectangular area under the large swing set if it is 1.5 times as long and 1.5 times as wide as the area under the small swing set?
High School Task Set

Task 1: Locate the incenter of the following triangle:

Task 2: You have been asked to place a warehouse so that it is an equal distance from the three roads indicated on the following map. Find this location and show your work.
Handout 2.3.1

**Focusing on Mathematical Statements**

Read the statements below about rectangles (written by grade-level teachers of mathematics), and then answer the questions below.

<table>
<thead>
<tr>
<th>Teacher Grade Level</th>
<th>Task: Write a statement about rectangles.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Special Education Elementary</td>
<td>If a figure has four right angles, then it is a rectangle.</td>
</tr>
<tr>
<td>7&lt;sup&gt;th&lt;/sup&gt;</td>
<td>If a shape is a rectangle, then it is a parallelogram with at least two right angles.</td>
</tr>
<tr>
<td>5&lt;sup&gt;th&lt;/sup&gt;</td>
<td>If a figure is a rectangle, then it is a parallelogram with at least one right angle.</td>
</tr>
<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt;</td>
<td>If a figure is a rectangle, then it is a four-sided polygon with four right angles.</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt;</td>
<td>If a figure is a rectangle, then it is a four-sided polygon with four right angles.</td>
</tr>
<tr>
<td>6&lt;sup&gt;th&lt;/sup&gt;</td>
<td>If a parallelogram is a rectangle, it has four right angles.</td>
</tr>
</tbody>
</table>

Are the statements precise?

Are the stated conditions necessary?

Are the stated conditions sufficient?
Handout 3.0.1

Standards for Mathematical Practice
MP2 and MP3

MP2. Reasoning abstractly and quantitatively

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complimentary abilities to bear on problems involving quantitative relationships: the ability to decontextualize — to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents — and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP3. Construct viable arguments and critique the reasoning of others

Mathematically proficient students understand and use stated assumptions, definitions and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed and — if there is a flaw in an argument — explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
Comparing Standards

CCSS for Mathematics Reasoning and Explaining Practices:

MP2. Reasoning Abstractly and Quantitatively
- Make sense of quantities and their relationships in problem situations
- Decontextualize — to abstract a given situation and represent it symbolically
- Contextualize — to probe into the referents for the symbols involved
- Create a coherent representation of the problem at hand
- Consider the units involved
- Attend to the meaning of quantities, not just how to compute them
- Know and flexibly use different properties of operations

MP3. Constructing Viable Arguments and Critiquing the Reasoning of Others
- Use stated assumptions, definitions, and previously established results in constructing arguments
- Make conjectures
- Build a logical progression of statements
- Analyze situations by breaking them into cases
- Recognize and use counterexamples
- Justify conclusions, communicate them to others and respond to the arguments of others
- Distinguish correct logic or reasoning from that which is flawed

Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.

Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Adapted from the CCSS Initiative Web site at http://www.corestandards.org/
CCSS for English Language Arts Anchor Standards:

This is a partial list of the ELA Anchor Standards, organized in a different order than that presented in the CCSS.

**Reading: Key Ideas and Details**

1. Read closely to determine what the text says explicitly and to make logical inferences from it; cite evidence when writing or speaking to support conclusions drawn from the text.

**Reading: Integration of Knowledge and Ideas**

8. Delineate and evaluate the argument and specific claims in a text, including the validity of the reasoning as well as the relevance and sufficiency of the evidence.

**Writing: Text Types and Purposes**

1. Write arguments to support claims in an analysis of substantive topics of texts, using valid reasoning and relevant and sufficient evidence.

**Writing: Production and Distribution of Writing**

5. Develop and strengthen writing as needed by planning, revising, editing, rewriting, or trying a new approach.

6. Use technology, including the Internet, to produce and publish writing and to interact and collaborate with others.

**Writing: Research to Build and Present Knowledge**

7. Conduct short as well as more sustained research projects based on focused questions, demonstrating understanding of the subject under investigation.

8. Gather relevant information from multiple print and digital sources, assess the credibility and accuracy of each source, and integrate the information while avoiding plagiarism.

9. Draw evidence from literary or informational texts to support analysis, reflection, and research.
Speaking and Listening: Comprehension and Collaboration

1. Prepare for and participate effectively in a range of conversations and collaborations with diverse partners, building on others' ideas and expressing their own clearly persuasively.

3. Evaluate a speaker's point of view, reasoning, and use of evidence and rhetoric.

Speaking and Listening: Comprehension and Collaboration

4. Present information, findings, and supporting evidence such that listeners can follow the line of reasoning and the organization, development, and style are appropriate to task, purpose, and audience.

Adapted from the CCR Anchor Standards for ELA on the CCSS Initiative Web page at http://www.corestandards.org/ELA-Literacy/CCRA/R
Partnership for 21st Century Skills (P-21)

Creativity and Innovation
Students use a wide range of techniques to create new and worthwhile ideas, elaborate, refine, analyze and evaluate their own ideas in order to improve and maximize creative efforts, and demonstrate originality and inventiveness, in both individual and group settings.

Critical Thinking and Problem Solving
Students reason effectively, use systems thinking and understand how parts of a whole interact with each other. They make judgments and decisions, and solve problems in both conventional and innovative ways.

Communication and Collaboration
Students know how to articulate thoughts and ideas effectively using oral, written and nonverbal communication. They listen effectively to decipher meaning, such as knowledge, values, attitudes and intentions, and use communication for a wide range of purposes in diverse teams and environments.

Handout 3.1.1

Odd and Even Survey

Defining Even

It is important in mathematics to give clear and logical explanations. Please take a few minutes to write down your ideas about the following:

1) What is the definition of an even number?

2) Explain why the sum of two even numbers is always even.
5th Grade

- A number that ends in 2, 4, 6, 8, 0. If you count by 2s, you will eventually make it to that number. It can be split in half equally.

- To find out which ones are even and which ones are odd, I think this one ... 2, 4, 6, 8. Every other one is going to be even.

- 4 is an even number because if you divide it by 2, it will have an equal set of numbers (see diagram below).
Odd and Even Survey: Student Responses

Middle School

- The definition of an even number is like they are multiplying by 2, because it goes to 2, 4, 6, 8 and etc. That's what I like about math.
- You skip the odd number and that's the even number.
- A number that can be evenly divided by half.

High School

- Usually not a prime number except two, and is not odd.
- An even number, such as two, can always be divisible and stays a composite number. The pattern of all numbers is odd, even, odd, even, etc. So every other number is even. If a number ends in an even number (2, 4, 6, 8 and technically 0) that number is automatically even.
- Because when you go on a date, you go with another person and there's two of you. And if you go on a double date, there's four of you. But if there is a third wheel because their date didn't show up; that's just awkward and odd ... no pun intended.
Handout 3.1.3

Sample Strategies for Differentiating Instruction

Consider the recommendations below to engage all students as they develop their mathematics reasoning skills:

1. Presenting information in multiple ways
2. Ensuring that students understand the text of problems
3. Using multiple resources and modes
4. Understanding flexibility in ways students respond
5. Maintaining the high cognitive demand of tasks/rigorous content
6. Providing diverse avenues of action and expression
7. Using performance-based assessments
8. Using language as a resource for learning not only as a tool for communicating, but also as a tool for thinking and reasoning mathematically
9. Focusing on mathematical discourse and academic language
10. Asking students to participate in mathematical reasoning by making conjectures, presenting explanations, and/or constructing arguments
11. Encouraging English learners to produce explanations, presentations, and discussions as they learn English
Lesson Plan for Day One

Grade Five Lesson Plan: What Makes a Number Odd or Even?

Note: Defining odd and even is not a grade five standard. The example lesson is meant to focus on the reasoning and explaining practices, not the standard itself.

Mathematical goals

This lesson is intended to:

- Assess how well students understand the concepts of odd and even. It will help to identify and support students who have difficulty in identifying whether a number is odd or even.

- Introduce students to making a viable argument about a number being odd or even.

Common Core State Standards

This lesson involves mathematical content in the standards from Operations and Algebraic Thinking:

- 2.OA.3. Determine whether a group of objects (up to 20) has an odd or even number or members (e.g., by pairing objects or counting them by 2s). Write an equation to express an even number as a sum of two equal addends.

- 4.OA.5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule “Add 3” and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

This lesson involves a range of mathematical practices, with emphasis on:

1. Reason abstractly and quantitatively

2. Construct viable arguments and critique the reasoning of others

3. Attend to precision
Handout 3.1.4 (Cont.)

Language goals

This lesson is intended to help assess whether students have the academic language to communicate their ideas about odd and even.

Vocabulary: Words in bold is essential vocabulary; non-bold is useful vocabulary.

| skeptic, represent, representation, representative, diagram, group, members, pair, paired, pairing, sum, addends, odd, even |

Phrases:
- odd (even) number of members in a group
- number sentence
- by pairing objects
- making twos
- making pairs

Structure of the Lesson

Students are introduced to the main question to work on: “Is 50 odd or even? How do you know?”

Students think about the question independently for a minute and then work with a partner to make an argument for 50 being an odd or even number.

During the lesson, students work in pairs or threes to justify their thinking to each other and to create a representation showing that 50 is either odd or even.

In a whole-class discussion, students show their different representations of the number 50 and explain their answers.

Time needed (approximate and will vary depending on needs of class)

5 minutes to introduce the task
1 minute to work independently
10 minutes to work with partner
5 minutes to create representation
15 minutes to discuss with whole class
<table>
<thead>
<tr>
<th>Lesson Structure</th>
<th>Instructions/Questions to Students</th>
<th>Instructions/Questions for teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>INTRO</strong></td>
<td><strong>The Rules of the Game (socio-mathematical norms that promote learning) [Whole class 5 mins]</strong></td>
<td></td>
</tr>
<tr>
<td>Purpose/Goal</td>
<td>Errors are gifts, because they promote discussion. The answer is important, but it is not the whole math! Ask questions until it makes sense. Think with language and use language to think. Multiple strategies . Multiple representations.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The purpose of today's lesson is for me to learn what you know about odd and even. I am a skeptic [do you know what a skeptic is?], your job is to convince me and convince the class [with words, diagrams, pictures, examples, non-examples, different representations] that you know the answer to this question.</td>
<td></td>
</tr>
<tr>
<td><strong>INTRO</strong></td>
<td><strong>The problem [Whole class 5 mins]</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Is 6 odd or even? How do you know?</td>
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<tr>
<td></td>
<td>We are going to watch a movie of a student working on that question. Then, you will have time to think about whether you agree or disagree with the student in the movie and explain, with a partner, why. But, remember you have to be convincing—you have to convince me and your classmates.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Do you understand what I am asking you to do?</td>
<td></td>
</tr>
<tr>
<td><strong>INTRO</strong></td>
<td><strong>Solo Thinking... [Individual 1 min.]</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>This is your time to think, to make sense of the problem</td>
<td></td>
</tr>
<tr>
<td></td>
<td>What am I asking you to do?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>What is six?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>What are ways to represent six?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>You won't have time to finish solving the problem but you will need to explain your way of thinking about the problem to your partner and possibly the class later. Now we need silence with pencil in hand, paper on table.</td>
<td></td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>
### Handout 3.1.4 (Cont.)

<table>
<thead>
<tr>
<th>Lesson Structure</th>
<th>Instructions/Questions to Students</th>
<th>Instructions/Questions for teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Answering the question</strong>&lt;br&gt;(pairs, table 10 min.)&lt;br&gt;Work in pairs to answer the question. Show your thinking. Remember, Alice can't see what you are pointing to when you point to something inside your head. Write something down, draw something that will help Alice see what you are thinking. Use with words, diagrams, pictures, examples, non-examples, different representations.</td>
<td>Think-Pair-Share prompt social norms of listening to each other by asking Alice what Alesia thinks. Model curiosity about and respect for the thinking of others try to identify the different ways of thinking in the class. Preserve the differences. Start thinking about who can present each way of thinking. You will only have time for 3.</td>
<td>Insist on replacing hand waving with drawing diagrams, writing things down, using labels, using the language of mathematics to express yourself! (This is extremely valuable for developing fluency with expressions, diagrams and other representations).</td>
</tr>
<tr>
<td><strong>Discussing</strong>&lt;br&gt;Prepare a presentation&lt;br&gt;(pairs 5 min.)&lt;br&gt;With your partner make a poster to help others in the class understand your thinking.</td>
<td>The preparation should be a focused team effort by partners. Once the culture of partner work has been established, partners can present to other partners at tables, and then tables can prepare for whole class. Preparation means making a poster or equivalent that can help make the thinking clear to the class during the student presentations.</td>
<td>That someone is listening to you and wants to know what you think is an almost irresistible motivation to talk. This is especially important for students who shy away from mathematics.</td>
</tr>
<tr>
<td>Prep to present&lt;br&gt;(pairs 5 min.)&lt;br&gt;Decide with your partner who is going to say what.</td>
<td></td>
<td></td>
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</tbody>
</table>
### Handout 3.1.4 (Cont.)

<table>
<thead>
<tr>
<th>Lesson Structure</th>
<th>Instructions/Questions to Students</th>
<th>Instructions/Questions for teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td>Presentations</td>
<td>Orchestrating discourse: Please listen carefully to ______ &amp; _______ Do they have a clear answer? Do they convince you?</td>
<td>Whole class listens to and interacts with selected presenters you model what someone does who wants to understand the other person's thinking Especially important when they are going through something that might be confusing</td>
</tr>
<tr>
<td>[3-5 Pairs 15 min.]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**DISCUSSING**
<table>
<thead>
<tr>
<th>Lesson Structure</th>
<th>Instructions/Questions to Students</th>
<th>Instructions/Questions for teacher</th>
</tr>
</thead>
</table>
| **Closure** (Whole class 10 min.) | What did we learn today?  
Who can summarize? What do we need to learn next?  
Feedback:  
What part of the lesson did you like the most?  
What part of the lesson was confusing?  
What questions do you still have? | praise good habits. "Nice table. Easy for me to understand."  
Also highlight high value content, "What's did she just use to do that? Distributive property. 5 point bonus. thank you".  
thank each presenter by name  
Interact with presenters, engage whole class in questions  
Object and focus is for all to understand thinking of each, including approaches that didn't work  
slow presenters down to make thinking explicit draw with marker correspondences across approaches  
Converge on mathematical target of lesson |
**Handout 3.2.1**

**Taxonomy of Questions in Mathematical Discourse:**
Questions and Responses

(adapted from work by Edith Prentice Mendez; Mitchell Nathan and Suyeon Kim)

<table>
<thead>
<tr>
<th>Question Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>External authority</td>
<td>Answers are attributed to someone else, teacher, parent, or “I just knew it”.</td>
</tr>
<tr>
<td>Recall – What is it?</td>
<td>Knowledge produced from memory (e.g., facts, calculations, definitions).</td>
</tr>
<tr>
<td>Justification – Why is it true?</td>
<td>Provides evidence for and against the claim. Relates concepts to situations, new concepts, concept to question.</td>
</tr>
<tr>
<td>Generalizations – Is it always true?</td>
<td>Communicates reasoning about commonalities in patterns, procedures, structures, and relationships.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Questions</th>
<th>Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>External Authority</td>
<td>Any question.</td>
</tr>
<tr>
<td>Confirm</td>
<td>Last year’s teacher told me.</td>
</tr>
<tr>
<td>Do you agree? Disagree?</td>
<td>Yes / no</td>
</tr>
<tr>
<td>Is it …. or ….?</td>
<td>Thumbs up / down</td>
</tr>
<tr>
<td>Recall</td>
<td>How many….? What did … say? What is…</td>
</tr>
<tr>
<td>How many….? What did … say? What is…</td>
<td>It is …</td>
</tr>
<tr>
<td>How are they the same? Different?</td>
<td>The answer is ….</td>
</tr>
<tr>
<td>And then what did you do?</td>
<td>First I … then I… and then…</td>
</tr>
<tr>
<td>Explain</td>
<td>They are … because ….</td>
</tr>
<tr>
<td>How did you ..? Can you explain ..?</td>
<td>If … then… because …</td>
</tr>
<tr>
<td>How are they the same? Different?</td>
<td>If … so … because…</td>
</tr>
<tr>
<td>And then what did you do?</td>
<td>So it would be …. because…</td>
</tr>
<tr>
<td>Justify</td>
<td>Why would you…? Why does that work?</td>
</tr>
<tr>
<td>Is there another way?</td>
<td>If … then… because …</td>
</tr>
<tr>
<td>What do you think?</td>
<td>If … so … because…</td>
</tr>
<tr>
<td>Generalize</td>
<td>Sentences will be followed with justifications.</td>
</tr>
<tr>
<td>(often combined with justification)</td>
<td>What I’m noticing is ….</td>
</tr>
<tr>
<td>Why? Does it always work? Is there a rule?</td>
<td>It will always….</td>
</tr>
<tr>
<td></td>
<td>Anytime ….</td>
</tr>
</tbody>
</table>
Handout for 3.2.2

Level 3:
Noticing Patterns Across Multiple Examples

Brandon: An even number is a number that ends in 0, 2, 4, 6 or 8. It can be split in half equally. If someone counts by twos, they will land on an even number.

Chris: An odd number is a number that ends in 1, 3, 5, 7 or 9. It cannot be split equally, because there would be a remainder. If someone counts by twos, they would not make it to an odd number.

Nathan: 2610 ends in 0. If someone counts by twos, that person would get to 2610. It can be split in half equally. If you add 1305 and 1305, you would get 2610. So 2610 is even.

Chris: 187 ends in 7. If someone counts by twos, they won’t get to that number. It cannot be split in half equally. 187 divided by 2 is 93 R-1. 93 has a remainder, so 187 is odd.

Brandon: 294 ends in 4. If someone counts by twos, they would get to this number. It can be split in half equally. 294 divided by 2 is 147, so 294 is even. This (representation above) represents the model of the multi-links that we built. Then the information we got from the graph, this graph. We recorded it onto this paper to help us understand which numbers were odd and even.

Teacher: Questions?

Tristin: Remember when you said it was 1, 3, 5, 7 or 9 on the back it’s odd? How did you guys know that?

Chris: How did we know if it’s …

Tristin: How did you figure out that if you put one of those numbers on the back, it would be odd?

Teacher: Good question, Tristin.

Chris: He’s asking us how did we know that? We would know, because we tested it out by dividing, like division. We just picked any random number by putting an odd number in the one’s place. We divided it and ended up with a remainder.
Engaging Diverse Learners

Many students need additional supports when presented with the tasks of explaining and justifying. Consider the recommendations below to engage all students as they develop their mathematics reasoning skills:

**English learners:**

“Instructional recommendations for teaching the Common Core to English learners include a focus on English learner’s mathematical reasoning; a shift to focus on mathematical discourse practices, moving away from simplified views of language, and support for English learners as they engage to complex mathematical language” (Moschkovich, 2012).

“Scaffolding should be provided through the use of multiple representations including choice of texts and tools (dictionaries, glossaries), teaching of key vocabulary, visual representations, models, strategic questioning, and coaching” (Santos, et al., 2012).

“Classroom environments that make ample use of the multiple modes — pictures, diagrams, presentations, written explanations, and gestures — afford English learners the means first to understand the mathematics with which they are engaged and second to express the thinking behind their reasoning and problem solving” (Driscoll, et. al. 2012).

**Students receiving special education services:**

“... strategies support student engagement by presenting information in multiple ways and allowing for students to access and express what they know in a variety of ways” (McNulty & Gloeckler, 2011).

“... a scientifically valid framework for guiding educational practice that provides flexibility in the ways information is presented, in the ways that students respond or demonstrate knowledge and skills, and in the ways students are engaged” (CCSS Initiative, 2010).
Handout for 3.3.1

Identifying Flaws Through Discourse

Jessica: You said, "If you group it into twos, 9 is odd, but if you group it into threes, it is even." But, I'm getting confused, because you say that it is even, and then you agree with the odd.

Chris: I don't agree that the number 3 decides if it's even or odd. I think the number 2 decides, because 2 is always even. If you count by twos, you would just get an even number, and you'll never land on an odd.

Gabby: The number $3 + 3 + 3$. It's not even, but it's equal.

Teacher: You just said something really important. What did you say about it's not even but equal?

Tristin: When you count by threes it goes odd (3), even (6), odd (9), even (12), until you stop.

Daniela: But you're saying that 3 is odd, and 6 is even, and 9 is odd. So 3, even though the next one, 6 is even, you start with odd (3).

Tristin: 'Cause usually when you count by fours, it's even: 4, 8, 12, 16, 20 ... Those are all even. Then 3, 6, 9, 12, 15 is odd, even, odd, even. Because usually like 2, 4, 6, 8 is all even. It just keeps going in a pattern.

Daniela: Yep, well every time you count by twos, it's even. But, the rest of the numbers are switching: odd, even, odd, even.

Max: I think what it's trying to say is that if it's 3, 3, and 3, those are all equal numbers but 3 is odd.

Daniela: But what about what Amity said (that) if you go to one group and it's only 3 then it's odd?

Max: It doesn't matter. Wouldn't you agree that 3, 3, and 3, it'd be even?

Daniela: The groups have the same amount.

Max: Exactly! 9 is both even and odd.

Chris: But like Tristin said, it goes in a pattern: odd, even, odd, even...and, if one number was both, it would break the pattern. I don't think a number can be both even and odd.

Teacher: Why not?

Chris: Because an odd number like 3, you can't split it in half equally. 'Cause like one team would have two, and the other would have one.

Daniela: It's kind of like how Mr. Asturias began with 50 and then the argument, "Is it odd or even?" begins.
Flower Arranging

This problem gives you the chance to:
• figure out how many flowers are in an arrangement
• divide a number into parts in order to satisfy given conditions

Tim’s grandmother loves flower arranging.

She always uses an odd number of each flower in each arrangement.

1. The arrangement she is making today has tulips, roses, and lilies. Tim’s grandmother uses 9 flowers in all.

There are more tulips in the arrangement than there are roses, and there are more roses than lilies.

How many tulips are there? 

How many roses are there? 

How many lilies are there? 

Explain how you figured this out.
2. The next arrangement Tim’s grandmother makes also has tulips, roses, and lilies. She uses 11 flowers in all.

As before, she always uses an odd number of each flower in each arrangement.

There are more tulips in the arrangement than there are roses, and there are more roses than lilies.

How many tulips are there?  
How many roses are there?  
How many lilies are there?  

Explain how you figured this out.

______________________________________________

______________________________________________

______________________________________________

______________________________________________
Handout 3.4.2

Brandon’s Flower Arrangement

Flower Arranging
The problem gives you the chance to:
• Figure out how many flowers are in an arrangement
• Divide a number into parts in order to satisfy given conditions.

Tim’s grandmother loves flower arranging.
She always uses an odd number of each flower in each arrangement.
1. The arrangement she is making today has tulips, roses, and lilies.
   Tim’s grandmother uses 9 flowers in all.
   There are more tulips in the arrangement than there are roses,
   and there are more roses than lilies.
   How many tulips are there? 5 tulips
   How many roses are there? 3 roses
   How many lilies are there? 1 lily

   Explain how you figured this out.
   I started with 9. Write all odd numbers to 9. Try different ways to make 9 with other
   odd #s. Find the right one. The largest #,
   in this case 5, goes here. Then 3, then 1.
2. The next arrangement Tim’s grandmother makes also has tulips, roses, and lilies. She uses 11 flowers in all.

As before, she always uses an odd number of each flower in each arrangement.

There are more tulips in the arrangement than there are roses, and there are more roses than lilies.

How many tulips are there?

How many roses are there?

How many lilies are there?

Explain how you figured this out.

First, I took odd numbers and added them in many ways. When I finally got a sum of 11, I drew something to show the corresponding number of each flower in the vase. I took the highest number in the equation, 7, and put it as the tulip, since it says there are more tulips than the other types of flowers. Then I put down 3 for roses and 1 for lilies.
Partners Sharing

In the following dialogue Leslie has taught her strategy to Stephanie and they share their method with the teacher. Look for points where the girls correct themselves or refine their argument as they are talking.

Teacher: Tell me what you have going on here.

Stephanie: I grabbed 9 pencils for the roses, 9 erasers for the tulips, and 9 pens for the lilies. Then I lined up all the odd numbers.

Leslie: So, I put: 1, 3 …

Stephanie: 5, 7 and 9.

Leslie: So, we’re going to pretend like that this box is like the vase for the … and she’s going to try to put in each flower into the rose, I mean into the vase, so that we have like odd numbers. So, try.

Stephanie: So, first I’m going to start with the roses.

Leslie: There are more tulips than roses.

Stephanie: Nine pencils for the roses. These are the tulips (pointing to the 9 erasers).

Leslie: So we’re going to put 5 tulips. 1, 2, 3, 4, 5. And since the tulips have to be bigger than the roses, we’re going to put …

Stephanie: 3 roses.

Leslie: Then we’re going to put 1 …

Stephanie: Tulip.

Leslie: 1 lily. Because there are more roses than lilies. Then we’re going to count it up, so it’s going to be 1, 2, 3, 4, 5, 6, 7, 8, 9.

Stephanie: 9
I split nine in half and I have to make it odd so four can't work because it's even. That is why five work. 5 are the tulips because the tulip are more than the roses (3), and there are more roses than lilies (1).

Strategy #1

I followed the strategy to use odd numbers only and there could only be nine flowers altogether. I split 9 in half to make 5. I left 5 along because it is already odd. So I took 4. I can't split it in half because it would still be even so I took 1 and 3 out of 4.

Odd is when you put 2 in group(s) there is 1 left over. Even is when you put 2 in group(s) there is no left over.

<table>
<thead>
<tr>
<th>1 3 5 7 9 11</th>
<th>odd</th>
<th>2 4 6 8 10</th>
<th>even</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

By Daniel, Michael, David, Josue
Handout 3.4.4

Student-Generated Math Challenge

Max: Along the way we discovered that theories have been tested; some were successful and some were not. A successful one was even + even = even and odd + odd = even, but even + odd = odd.

- even plus even: $4 + 4 = 8$ ($2, 4, 6, 8$) even
- odd plus odd: $3 + 9 = 12$ ($2, 4, 6, 8, 10, 12$) even
- even plus odd: $4 + 9 = 13$ ($2, 4, 6, 8, 10, 12, 14$) odd

Spencer: The theory we had was that if you plus a number plus itself, no matter what, you will get an even number.

- $1785 + 1785 = 3570$ ... even!
- $1853 + 1853 = 2706$ ... even!

The other theory we had was that if you add an even plus an odd number, you will get an odd number no matter what.

- $19 + 24 = 43$
- $18 + 17 = 35$

Haley:

- odd + odd = even.
  - $1 + 1 = 2$
  - $3 + 3 = 6$
  - $5 + 5 = 10$
  - $7 + 7 = 14$
  - $9 + 9 = 18$

Conjecture: For odd and odd will always equal to even.

- even + even = even
  - $0 + 0 = 0$
  - $2 + 2 = 4$
  - $4 + 4 = 8$
  - $6 + 6 = 12$
  - $8 + 8 = 16$

Conjecture: For even and even will always equal to even.

- odd + even = odd
  - $1 + 0 = 1$
  - $3 + 0 = 3$
  - $5 + 0 = 5$
  - $7 + 0 = 7$
  - $9 + 0 = 9$

This student-generated challenge fits the descriptors of Ball's reasoning community, stating how public knowledge (e.g., the definition of odd and even) becomes the foundation for new knowledge under construction (e.g., developing rules for adding odd and even numbers).
Standards for Mathematical Practice

MP 4. Model with mathematics
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation, as well as reflect on whether the results make sense, and possibly improve the model if it has not served its purpose.

MP 5. Use appropriate tools strategically
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include: paper and pencil, concrete models, a ruler, protractor, calculator, spreadsheet, computer algebra system, statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a Web site, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.
Tiling Pool Problem

Tat Ming is designing square swimming pools. Each pool has a square center that is the area of the water. Tat Ming uses blue tiles to represent the water. Around each pool there is a border of white tiles. Below are pictures of the three smallest pools that he can design with the blue tiles for the interior and white tiles for the border (Ferrini-Mundy, et al., 1997).

Grades K–2

1. For each square pool, sort the tiles into blue tiles for the water and white tiles for the border.

2. Count how many tiles are in each pile. Are there more blue tiles than white tiles?

3. How many tiles are in the next largest pool? Check your answer by building the square.

4. Describe your methods for counting the different tiles. What patterns do you see?

Grades 3–5

1. Build the first three pools and record the data in a table. Extend the table for the next two pools. How do you know your answers are correct?

2. If there are 32 white tiles in the border, how many blue tiles are there? Explain how you got your answer.

3. If there are 36 blue tiles, how many white tiles are there? Explain how you got your answer.

5. Can you make a pool with 12 blue tiles? Explain why our why not.

6. In each of the first three pools, decide what fraction of the square’s area is blue for the water and what fraction is white for the border? What patterns do you see? What fractions will occur in the next two rows of the table? How do you know that your answers are correct?

7. Below is a picture of Salina’s backyard. If each tile has a side length of 10 centimeters, what is the largest pool someone could put in Salina’s backyard?
Grades 6–8

1. Make a table showing the numbers of blue tiles for water and white tiles for the border for the first six pools.

2. What are the variables in the problem? How are they related? How can you describe this relationship in words?

3. Make a graph that shows the number of blue tiles in each pool. Make a graph that shows the number of white tiles in each pool.

4. As the number of pool tiles increases, how does the number of white tiles change? How does the number of blue tiles change? How does this relationship show up in the table and in the graph?

5. Use your graph to find the number of blue tiles in the seventh pool.

6. Can there ever be a border for a pool with exactly twenty-five white tiles? Explain why or why not.

7. Below is a picture of Salina’s backyard. What is the largest pool Salina can build in her backyard? (Note: This question raises the problem to Level 3 modeling).
1. What are the variables in this situation? What quantities are changing?

2. How are the variables related? As one variable increases, what happens to the other variable?

3. How can you represent the relationship between: a) the pool number and the white tiles, b) the pool number and the blue tiles, and c) the pool number and the total number of blue and white tiles, using words, tables, graphs, and symbols?

4. Are any of the relationships in question 3 functions? If so, what types of functions are they (e.g., linear, quadratic, exponential)? How do you know? Explain.

5. What if the pool is a non-square, rectangle? Explain in words, with numbers or tables, visually, graphically, and with symbols, the number of tiles that will be needed for pools of various lengths and widths. (Note: This question raises the problem to Level 3 modeling.)
Handout 4.2.1

Quotes on Modeling

**Early Grades:** “Children naturally attempt to model the action or relationships in problems. They first directly model the situations or relationships with physical objects. Children’s solution strategies are first fairly exact models of problem actions or relationships. Counting and Direct Modeling strategies are simply specific instances of the fundamental principle of modeling. It is helpful to think of these strategies as attempts to model problems rather than as a collection of distinct strategies. The conception of problem solving as modeling not only serves as a basis for understanding children’s strategies for solving addition, subtraction, multiplication, and division problems, it also can provide a unifying framework for thinking about problem solving in the primary grades.”

Carpenter, et al., 1999

**Middle Grades:** “The habit of problem posing, creating representations, explaining connections, and testing and checking are central to the development of interesting and new mathematics and applications. Real world applications often involve many variables, incomplete information, multiple methods of solution, and answers that vary according to the assumptions, and simplifications made and approach taken. Encounters with such settings dispel students’ notion that the trademark of mathematics is the exactness and uniqueness of results. Rather recognition of underlying structure and abstraction become dominant features of the discipline. Teachers must help students become comfortable with uncertainty while striving for clarity in their descriptions and analyses. Students must accept that creativity and clear communication are part of active learning and discovery. Lastly, they must be curious and willing to take risks. Successful students in traditional math classes are rewarded for speed and technical accuracy. A different type of confidence is required when they begin posing problems with no immediate clear method of solution and no guarantee that a solution can be found”.

Abrams, 2001

**High School:** “Mathematical modeling is a form of problem solving. A mathematical model should be mathematically accurate and portray a situation in the real world. Modeling considers solutions in terms of the situation that spawned the mathematical problem. A student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. At the high school level, teachers are expected to provide opportunities that allow students to examine how multiple representations support different ways of thinking about and manipulating mathematical objects. Students need an opportunity to practice converting among different representations for a given situation to create flexibility with modeling. These opportunities should emphasize selection of a certain representation for a mathematical situation based on what information the representation needs to convey.”

Christinson, et al., 2012
Examples of Grades K–2 Modeling Activities

The problems below represent Level 1 modeling tasks:

**Problem 1:** Connie had 5 marbles. Juan gave her 8 more marbles. How many marbles does Connie have altogether?

**Problem 2:** Connie had 13 marbles. She gave 5 to Juan. How many marbles does Connie have left?

**Problem 3:** Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?

**Problem 4:** Connie has 13 marbles. Juan has 5 marbles. How many more marbles does Connie have than Juan?

**Problem 5:** Gene has 4 tomato plants. There are 6 tomatoes on each plant. How many tomatoes are there altogether?

**Problem 6:** Ellen walks 3 miles an hour. How many hours will it take her to walk 15 miles?

**Problem 7:** Jan bought 7 pies. He spent a total of $28. If each pie cost the same amount, how much did one pie cost?

**Problem 8:** The giraffe in the zoo is 3 times as tall as the kangaroo. The kangaroo is 6 feet tall. How tall is the giraffe?

For additional practice, refer to *Children’s Mathematics Cognitively Guided Instruction* (Carpenter, 1999), a publication that includes videos showing students engaging in these tasks.
Examples of Grades 3–5 Modeling

This problem represents a Level 2 modeling task.

Problem 1: Austin’s mom is making healthy snacks for the students in his class. She finds a recipe on the Internet with the following ingredients. However, she didn’t see how many servings the recipe will make. What are some reasonable estimates on how much ingredients Austin's mom needs to make enough for all the students in the class to get a healthy snack? What information would be helpful to know?

http://kidshealth.org/kid/recipes/recipes/snack_mix.html#cat20229

Prep time: 5 minutes

What you need:
- 1 cup whole grain cereal (squares or Os work best)
- ¼ cup dried fruit of your choice
- ¼ cup nuts, such as walnut pieces, slivered almonds, or pistachios
- ¼ cup small, whole-grain snack crackers or pretzels

Equipment and supplies:
- Large bowl
- Measuring cups
- Large spoon

What to do:
- Measure out ingredients
- Combine in large bowl

This problem represents a Level 3 modeling task:

Problem 2: Lamar wants a new toy truck that sells for $25. He has $3 now. Create a plan that would help Lamar buy his truck three weeks from today.
Handout 4.3.1

Quotes on Tools

Grades K–2: “Educational research indicates that the most valuable learning occurs when students actively construct their own mathematical understanding. One way to facilitate this is to provide opportunities for children to explore, develop, test, discuss, and apply ideas. Extensive and thoughtful use of physical materials, particularly in the primary grades, is conducive to the concrete kinds of learning that lay a satisfactory foundation for the development of this mathematical understanding”

Johnson, 2012

Grades 3–5: “Manipulative materials in teaching mathematics to students hold the promise that manipulatives will help students understand mathematics. At the same time, as with any "cure", manipulatives hold potential for harm if they are used poorly. (No matter what the grade level of the students, the sole use of manipulatives should not be for modeling procedures; instead manipulatives should be made available as tools for problem solving.) Manipulatives that are improperly used will convince students that two mathematical worlds exist — manipulative and symbolic. All mathematics comes from the real world. Then the real situation must be translated into the symbolism of mathematics for calculating. For example, putting three goats with five goats to get eight goats is the real world situation but on the mathematics level we say 3+5 = 8 (read three add five equals eight). These are not two different worlds but they are in the same world expressing the concepts in different ways.”

Teaching Today, 2012

Grades 6–12: “The effectiveness of hands-on learning does not end in 5th grade. Research indicates that students of all ages benefit by being introduced to mathematical concepts through physical exploration. Planning lessons that proceed from concrete to pictorial to abstract representations of concepts makes content mastery accessible to students of all ages. With concrete exploration (through touching, seeing, and doing), students can gain deeper and longer-lasting understandings of math concepts.”

Teaching Today, 2012
Handout 4.4.1

Using Tools: K–2 Task and 3–5 Task

Grades K–2 Task: What's Missing?

Tools: Colored counters

Show the students six counters. Ask each student to close his/her eyes. Hide some of the counters under a sheet of heavy paper. When the student opens his/her eyes, s/he determines how many counters are hidden based upon the number of counters still showing.

Grades 3–5 Task: Battle Ship Using Grid Paper

Tools
Grid paper and colored pencils; one color for the ships and another for explosions on their ships and their enemy’s ships. This is how they will keep track of what ordered pairs have been called.

Setup
Students begin by folding the grid paper in half. They need to draw coordinate axes on both the top half and the bottom half and label the x- and y-axes with the numbers 1–10 on each axis. The students will need to illustrate 5 ships on ordered pairs and label the ordered pairs. They should draw:

- Two ships sitting on 2 ordered pairs
- One ship sitting on 3 ordered pairs
- One ship sitting on 4 ordered pairs
- One ship sitting on 5 ordered pairs

Remind the students that the bottom half of the grid paper contains their boats (or “Navy”) and the top half has their opponent’s boats.

Actions
Students play in pairs sitting opposite each other and take turns calling out ordered pairs. Players should keep a list of the ordered pairs they call out written in (x,y) form on a piece of paper that both players can see. This will ensure there is no disagreement about what has been called (it is common for students to transpose the coordinates).

Then they are to mark the ordered pair they call out on the top coordinate plane. They should mark in black if they missed and red if they hit their opponent’s boat. On the bottom half of the grid paper they are to color black for the ordered pairs their opponent calls out and color red for the ordered pairs that hit their ship.
Middle School Task: Triangular Tables

**Tools**
Pattern blocks

A classroom has triangular tables. There is enough space at each side of a table to seat one student. The tables in the class are arranged in a row (as shown in the picture below).

How many students can sit around one table? Around a row of two tables? Around a row of three tables?

Find an algebraic expression that describes the number of students that can sit around a row of \( n \) tables. Explain in words how you found your expression.

If you could make a row of 125 tables, how many students would be able to sit around it?

If there are 26 students in the class, how many tables will the teacher need to seat all of them around a row of tables?
High School Task: Kitchen Floor Tiles

Tools
Colored Tiles

Fred decides to cover the kitchen floor with tiles of different colors. He starts with a row of four tiles of the same color. He surrounds these four tiles with a border of tiles of a different color (Border 1). The design continues as shown below:

Dina writes, $t = 4 \cdot (b-1) + 10$ where $t$ is the number of tiles in each border and $b$ is the border number. Explain why Dina’s equation is correct.

Emma wants to start with five tiles in a row. She reasons, “Dina started with four tiles and her equation was $t = 4(b - 1) + 10$. So if I start with five tiles, the equation will be $t = 5(b - 1) + 10$. Is Emma’s statement correct? Explain your reasoning.

If Emma starts with a row of $n$ tiles, what should the formula be?
Standards for Mathematical Practice

MP7 and MP8

MP7. Look for and make use of structure
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see $7 \times 8$ equals the well-remembered $7 \times 5 + 7 \times 3$ in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers $x$ and $y$.

MP8. Look for and express regularity in repeated reasoning
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1,2)$ with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.
Consecutive Sums

Some numbers can be written as a sum of consecutive positive integers:

\[
\begin{align*}
6 &= 1 + 2 + 3 \\
15 &= 4 + 5 + 6 \\
   &= 1 + 2 + 3 + 4 + 5
\end{align*}
\]

Which numbers have this property? Explain.

Let’s look at what might be expected of students at each grade span when working on this problem.

K–2 Example

Write the first 10 numbers as a sum of other numbers. Which of these sums contain only consecutive numbers?

3–5 Example

Some numbers can be written as a sum of consecutive positive integers, for example:

\[
\begin{align*}
6 &= 1 + 2 + 3 \\
15 &= 4 + 5 + 6 \\
   &= 1 + 2 + 3 + 4 + 5
\end{align*}
\]

In small groups, find all numbers from 1–100 that can be written as a consecutive sum. Look for patterns as you work. Conjecture which numbers can and which cannot be expressed as a consecutive sum. How can some of the sums be used to find others?
Handout 5.1
Consecutive Sums (cont.)

6–8 Example

Some numbers can be written as a sum of consecutive positive integers, for example:

\[
\begin{align*}
6 & = 1 + 2 + 3 \\
15 & = 4 + 5 + 6 \\
& = 1 + 2 + 3 + 4 + 5
\end{align*}
\]

Which numbers have this property?

Sally made this conjecture: “Powers of 2 cannot be expressed as a consecutive sum.”

Agree or disagree and explain your reasoning.

9–12 Example

Some numbers can be written as a sum of consecutive positive integers, for example:

\[
\begin{align*}
6 & = 1 + 2 + 3 \\
15 & = 4 + 5 + 6 \\
& = 1 + 2 + 3 + 4 + 5
\end{align*}
\]

Exactly which numbers have this property?

When investigating this problem, Joe made the following conjecture: “A number with an odd factor can be written as a consecutive sum and the odd factor will be the same as the number of terms.” Agree or disagree with this statement and explain your reasoning.
Handout 5.2.1

Square Tiles

Tiles are arranged to form pictures like the ones below:

![Pictures of square tiles]

A. Find a direct formula that enables you to calculate the number of square tiles in Picture “n.” How did you obtain your formula?

If the solution has been obtained numerically, is there a way to explain your formula from the figures?

B. How many squares will there be in Picture 75? Explain.

C. Can you think of another way of finding a direct formula?

D. Two 6th graders came up with the following two formulas:

Kevin’s direct formula is: \( T = (n \times 2) + (n \times 2) + 1 \), where “n” means picture number and “T” means total number of squares.

Is his formula correct? Why or why not?

E. Melanie’s direct formula is: \( T = (n \times 2) + 1 + (n \times 2) + 1 -1 \), where “n” and “T” mean the same thing as in Kevin’s formula.

Is her formula correct? Why or why not?

F. Which formula is correct: Kevin’s formula, Melanie’s formula, or your formula? Explain.
K–2 Geometry Example

Consider what might make the task below easier for students in K–2 classrooms. What questions would you ask to reinforce the relationships among the pattern block pieces?

Brian made the animal picture below out of pattern blocks. Can you show how Brian made his animal figure using pattern blocks?

![Pattern Block Animal](image)

3–5 Geometry Example

While you work on the problem below, think about what questions you might ask students in grades 3–5 to assist them in finding the solution.

Two vertices of a triangle are located at (4,0) and (8,0). The perimeter of the triangle is 12 units. What are all possible locations for the third vertex? How do you know you have them all?

Source: Driscoll, 2007
Handout 5.3.2

**6–8 Geometry Example**

View the first 10 minutes of the Third International Math and Science Study (TIMSS) video on changing shape without changing area (you can view the entire clip after you complete the activity below):


Construct the land figure with the crooked boundary posed by the teacher to the class in the video above. Find a way to make the boundary straight without changing the area.

**9–12 Geometry Example**

Explore the proof of the Pythagorean Theorem presented in the dynamic NTCM link below:


Consider why this proof of the Pythagorean Theorem works and where structure plays a role in the proof.
K–2 Example: Grade 2 Task

Building Walls

Maya is using blocks to make a wall grow.

1. Draw the 4th wall below:

2. How many blocks are needed to make the 4th wall? ________ blocs

3. Tom wants to build the 5th wall. Tell Tom how to build the 5th wall.
   ____________________________________________________________
   ____________________________________________________________
   ____________________________________________________________

2nd Grade – 2008
Copyright ©2008 by Steck-Vaughn
All rights reserved.
4. How many blocks would it take to build the 7th wall? ______ blocks
   Show how you know that your answer is correct.

5. Tom and Maya’s teacher gave them 21 blocks and asked them to build
   a wall with all the blocks. Can they build a wall with exactly 21 blocks?
   Yes or No __________
   Why or Why not?
   __________________________________________________________
   __________________________________________________________
   __________________________________________________________
Handout 5.4.1 (Cont.)

Student Work for Building Walls

Student F

3. Tom wants to build the 5th wall. Tell Tom how to build the 5th wall.

- get ten blocks and add 1
- two blocks put on one block on the bottom right side then put the other block and put it in turn of the block that is behind it.

Student G

1. Draw the 4th wall below:

2. How many blocks are needed to make the 4th wall? 10

3. Tom wants to build the 5th wall. Tell Tom how to build the 5th wall.

- put seven blocks below and put the blocks on top.
Handout 5.4.1 (Cont.)

Student H

4. How many blocks would it take to build the 7th wall? 16
   blocks.
   Show how you know that your answer is correct.

5. Tom and Maya's teacher gave them 21 blocks and asked them to build a wall with all the blocks. Can they build a wall with exactly 21 blocks?
   Yes or No: Yes
   Why or Why not? Because the others walls built by 16 or lower.

How is Student F seeing the pattern? Does the student’s picture match the words? Where is the 10 in the pattern for the 5th stage?

Compare and contrast Student G’s work with Student F’s work. How many groups of two does Student G see in the 4th and 5th walls?

How will Student G’s drawing help him/her associate the number of the wall being built with the number of cubes needed to build that wall?

How does seeing this pattern begin to move the student toward a generalization?

How does Student H’s work illustrate progress toward making a generalization?

How does Student H use the structure of the visual pattern to find the number of cubes for the 7th wall?
Handout 5.4.2

3–5 Example: Grade 5 Task

**Hexagons in a Row**

This problem gives you the chance to:

- find a pattern in a sequence of diagrams
- use the pattern to make a prediction

Joe uses toothpicks to make hexagons in a row.

1 hexagon

6 toothpicks

2 hexagons

11 toothpicks

3 hexagons

16 toothpicks

4 hexagons

Joe begins to make a table to show his results.

<table>
<thead>
<tr>
<th>Number of hexagons in a row</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of toothpicks</td>
<td>6</td>
<td>11</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Fill in the empty spaces in Joe’s table of results.
2. How many toothpicks does Joe need to make 5 hexagons? ____________
   Explain how you figured it out.
   ________________________________________________________________
   ________________________________________________________________

3. How many toothpicks does Joe need to make 12 hexagons? ____________
   Explain how you figured it out.
   ________________________________________________________________
   ________________________________________________________________

4. Joe has 76 toothpicks.
   How many hexagons in a row can he make? ________________
   Explain how you figured it out.
   ________________________________________________________________
   ________________________________________________________________
   ________________________________________________________________
Student A

1. Fill in the empty spaces in Joe’s table of results. you can also multiply the amount of hexagons by 6 and subtract 1 toothpick for each shared toothpick (e.g. $2 \times 6 = 12$ shared + $12 - 1 = 11$)

2. How many toothpicks does Joe need to make 5 hexagons? 36 toothpicks

   Explain how you figured it out.
   The pattern is you add five toothpicks per hexagon. If 5 hexagons have 21 toothpicks, then 6 hexagons have 26 toothpicks. You could also multiply 5 by 6 and subtract 1.

3. How many toothpicks does Joe need to make 12 hexagons? 61 toothpicks

   Explain how you figured it out.
   All 5 did were multiply $6 \times 12$ (12 less than subtract 11 for the 11 shared toothpicks) / $72 - 11 = 61$ toothpicks

4. Joe has 76 toothpicks. How many hexagons in a row can he make?

   Explain how you figured it out.
   I multiplied 15 by 6 which equals 90.
   Then subtracted 14 for the 14 shared toothpicks / $90 - 14 = 76$ (I worked backwards)
2. How many toothpicks does Joe need to make 5 hexagons? Explain how you figured it out.

\[ 6 + 6 + 6 + 6 + 6 + 6 = 26 \]

3. How many toothpicks does Joe need to make 12 hexagons? Explain how you figured it out.

\[ 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 61 \]

4. Joe has 76 toothpicks. How many hexagons in a row can be made? Explain how you figured it out.

Count by 5's till we get to 70, then add 6, then you get 76.
Student C

4. Joe has 76 toothpicks.
   How many hexagons in a row can he make?
   Explain how you figured it out.

   You first know that the first hexagon needs 6 so you minus 76-6=70
   so you record you have 1.
   After you divide 5/70 now you know you have 15 because 5 goes in to 70 14 times after you
   add 14+1=15 hexagons

Student D

2. How many toothpicks does Joe need to make 5 hexagons?  \[ \square \] 26
   Explain how you figured it out.
   \[ \text{The pattern is add 5 more toothpicks for every hexagon} \]

3. How many toothpicks does Joe need to make 12 hexagons?  \[ \square \] 61
   Explain how you figured it out.
   \[ \text{I did 7x5 because I need 7 more hexagons than 5 then}
   \text{add 7x5 how many toothpicks do or make 5 hexagons} \]

4. Joe has 76 toothpicks.
   How many hexagons in a row can he make?
   Explain how you figured it out.
   \[ \text{I figured out by doing 76-61 (12 hexagons)} \]
   \[ 15 \text{ with 15 tooth picks I can make 1 more hexagon} \]
Handout 5.4.2 (Cont.)

Hexagons in a Row

How does Student A understand the structure of the pattern?

If Student A was in a later grade and could write a symbolic form for the pattern, what would the generalization for Student A’s pattern be?

How does Student A’s approach compare to Tamara’s from the video in section 5.3?

Student B visualizes the pattern in a different way from Student A. What is Student B’s approach?

What would be a symbolic representation for Student B’s visualization? Can you show it is equivalent to the symbolic form you got for Student A above?

How does Student C use his/her visualization scheme to work backward in part 4 of the problem? Can you explain the thinking? What is the inverse function being used by Student C?

Student D begins by making a table, as Tamara did in her post interview. How does the table help Student D determine the pattern? How does it help Student D figure out the answer to part 4 of the task?
Handout 5.4.3
6–8 Example: Grade 7 Task

Necklaces
This problem gives you the chance to:
• work with a sequence of bead patterns
• write a formula

Janice is making necklaces with colored beads. She makes them into square patterns like this:

1 square 2 squares 3 squares

1. Fill in the table showing the number of round and long beads needed.

<table>
<thead>
<tr>
<th>Number of squares</th>
<th>Long beads</th>
<th>Round beads</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Explain how you figured out how many long beads are needed to make 4 and 8 squares.

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________
3. Explain how you figured out how many round beads are needed to make 4 and 8 squares.

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

4. Janice uses 37 long beads to make some squares.
   a. How many squares does she make?
      Show your work.

   b. How many round beads will she need to make these squares? ________________

5. Write a rule or an algebraic formula for finding the total number of round and long beads. If Janice needs to make $n$ squares.

   ________________________________________________________________________
Handout 5.4.3 (Cont.)

Student Work for Necklaces

Student B

1. Fill in the table showing the number of round and long beads needed.

<table>
<thead>
<tr>
<th>Number of squares</th>
<th>Long beads</th>
<th>Round beads</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x+1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$12+1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$13$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3x+1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$24+1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$25$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3x+1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Explain how you figured out how many long beads are needed to make 4 and 8 squares.

I figured out a rule for finding the number of long beads, $3x+1$ (x represents the number of squares). I used my rule to find the number of long beads by substituting 4 for $x$ and then 8 for $x$.

3. Explain how you figured out how many round beads are needed to make 4 and 8 squares.

I found a rule to find the number of round beads, $2x+2$ (x is the number of squares). I used my rule to find the number of round beads by substituting 4 and 8 for $x$.

4. Janice used 37 long beads to make some squares.

a. How many squares does she make? 12 squares

Show your work:

$b \div 3 = \frac{314}{3}$

b. How many round beads will she need to make these squares? 26 round beads

$2 \div 2 + 2 = 26$

5. Write a rule or an algebraic formula for finding the total number of round and long beads, B, Janice needs to make a square.

$5n+3 = B \checkmark$

<table>
<thead>
<tr>
<th>$5+3$</th>
<th>$5+3$</th>
<th>$3+5+3$</th>
<th>$4+5+3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$13$</td>
<td>$18$</td>
<td>$23$</td>
</tr>
<tr>
<td>$5 \times 3$</td>
<td>$2 \times 3$</td>
<td>$3 \times 3$</td>
<td>$4 \times 3$</td>
</tr>
<tr>
<td>$40 \times 3$</td>
<td>$13$</td>
<td>$18$</td>
<td>$23$</td>
</tr>
</tbody>
</table>
2. Explain how you figured out how many long beads are needed to make 4 and 8 squares.

   I found the rule to figure out what number goes in a specific column. For instance, the adding increases by 2 for the 1st and columns as you go down the row. I did the same for the numbers in between 4 and 8 to make sure I have added the right number.

3. Explain how you figured out how many round beads are needed to make 4 and 8 squares.

   It's similar to the last question – the difference increases by one for the 1st and columns. 4 - 4 is 0, 7 - 6 is 1. The difference between the long beads and round beads increases with this rule. I did the same thing for the numbers between 4 and 8 to make sure I was subtracting the right number.

4. Janice used 37 long beads to make some squares.

   a. How many squares does she make?

   
   Show your work.
   
   37 long beads
   
   - 25 long beads
   
   = 12 long beads
   
   12 ÷ 3 = 4 (difference between the number of long beads in every row)
   
   + 8 (previous number of squares)

   b. How many round beads will she need to make these squares?

   24 round beads

5. Write a rule or an algebraic formula for finding the total number of round and long beads, B, Janice needs to make a squares.

   \[ N - 3(x + 4) = B \]

   0

   43 - 8 = 35
   48 - 9 = 39
   53 - 10 = 43
   58 - 11 = 47
   63 - 12 = 51
Student B is able to combine expressions to get the formula for the total number of beads in part 5. Why is trying different numbers in the formula a good habit of mind for students to have? Is this a proof?

Student C cannot discern the correct pattern. What strategy does this student use to find the numbers of beads? Why does this approach inhibit the student from being able to find a general formula for the number of beads?

Notice the naïve approach of Student D to the problem. How does this student determine the number of beads? What kind of rule does Student D get for part 5?

At this grade level we expect a direct formula. How can you help such a student move toward a direct formula?
Handout 5.4.4

9–12 Example: Grade 9 Integrated

Course 1 Task:

Sidewalk Patterns
This problem gives you the chance to:
• work with patterns
• work out the \(n\)th term of a sequence

In Prague some sidewalks are made of small square blocks of stone.
The blocks are in different shades to make patterns that are in various sizes.

1. Draw the next pattern in this series.

You may not need to use all of the squares on this grid.
2. Complete the table below.

<table>
<thead>
<tr>
<th>Pattern number, n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of white blocks</td>
<td>12</td>
<td>40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of gray blocks</td>
<td>13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total number of blocks</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. What do you notice about the number of white blocks and the number of gray blocks?

__________________________________________________________

4. The total number of blocks can be found by squaring the number of blocks along one side of the pattern.
   a. Fill in the blank spaces in this list.
   
   \[25 = 5^2 \quad 81 = \quad 169 = \quad 289 = 17^2\]

   b. How many blocks will pattern number 5 need?

   ___________________________________________

   c. How many blocks will pattern number \(n\) need?

   ___________________________________________

5. a. If you know the total number of blocks in a pattern you can work out the number of white blocks in it. Explain how you can do this.

   ___________________________________________

   ___________________________________________

   b. Pattern number 6 has a total of 625 blocks. How many white blocks are needed for pattern number 6? Show how you figured this out.

   ___________________________________________
Handout 5.4.4 (Cont.)

Student Work for Sidewalk Patterns

Student C

c. How many blocks will pattern number n need?

5. a. If you know the total number of blocks in a pattern you can work out the number of white blocks in it. Explain how you can do this.

You can figure out the number of white \( \times 0 \) blocks because you know the total blocks. How many white blocks are needed for pattern number 6? Show how you figured this out.

b. Pattern number 6 has a total of 625 blocks. How many white blocks are needed for pattern number 7? Show how you figured this out.

Student D

a. Fill in the blank spaces in this list.

25 = \( 5^2 \)  81 = \( 3^4 \)  169 = \( 13^2 \)  289 = \( 17^2 \)  81

b. How many blocks will pattern number 5 need?

441

(Previous number 4) \( ^2 \)

c. How many blocks will pattern number n need?

Student E

2. Complete the table below.

<table>
<thead>
<tr>
<th>Pattern number, n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of white blocks</td>
<td>12</td>
<td>40</td>
<td>( 4^2 )</td>
<td>un</td>
</tr>
<tr>
<td>Number of grey blocks</td>
<td>13</td>
<td>( 4^2 )</td>
<td>40</td>
<td>( 4^2 )</td>
</tr>
<tr>
<td>Total number of blocks</td>
<td>25</td>
<td>61</td>
<td>169</td>
<td>289</td>
</tr>
</tbody>
</table>

3. What do you notice about the number of white blocks and the number of grey blocks?

Notice that the number of grey blocks is always \( \frac{4}{2} \) the number of white blocks.

4. The total number of blocks can be found by squaring the number of blocks along one side of the pattern.

a. Fill in the blank spaces in this list.

25 = \( 5^2 \)  81 = \( 3^4 \)  169 = \( 13^2 \)  289 = \( 17^2 \)  81

b. How many blocks will pattern number 5 need?

441

c. How many blocks will pattern number n need?
Student C realizes that the number of grays is one more than the number of whites, but does not use the information to help inform his/her guess and check work. How could this relationship help narrow down the values?

Student D sees a pattern in the answers in 4a and uses that to write a recursive rule for the $n$th term. At the high school level, a direct formula is expected. How might you help Student D move from the recursive to a direct formula?

Student E has noticed the following interesting pattern: $5^2 + 4^2 = \text{the total number of gray tiles for stage 2}$; $7^2 + 6^2 = \text{the total number of gray tiles for stage 3}$. How might this pattern help you find an equation for 4c?

Student I notices that the number of whites in pattern number 3 is $21 \times 4$. Where does that number come from in the figure? How might it help in getting a generalization? What else do you notice in Student I’s work?
Handout 6.1.1

Self-Reflection Survey

Name ______________________________________________________

Common Core State Standards for Mathematical Practice

How confident are you in supporting all students to be successful in the following eight mathematical practices:

<table>
<thead>
<tr>
<th></th>
<th>Not Confident</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Very Confident</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make sense of problems and persevere in solving them</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Reason abstractly and quantitatively</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Construct viable arguments and critique the reasoning of others</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Model with mathematics</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Use appropriate tools strategically</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Attend to precision</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Look for and make use of structure</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Look for and express regularity in repeated reasoning</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
</tbody>
</table>
Handout 6.1.2

Post-Assessment

1. The design of the Common Core SMP was informed by:
   ☐ b. The National Council of Teachers of Mathematics (NCTM) Process Standards
   ☐ c) Both “a” and “b”
   ☐ d) None of the above

2. When students decontextualize a situation they apply the following mathematical practice:
   ☐ a) Construct viable arguments and critique the reasoning of others
   ☐ b) Use appropriate tools strategically
   ☐ c) Reason abstractly and quantitatively
   ☐ d) Attend to precision

3. When students use clear definitions in a discussion, they apply the following mathematical practice:
   ☐ a) Look for and make use of structure
   ☐ b) Use appropriate tools strategically
   ☐ c) Reason abstractly and quantitatively
   ☐ d) Attend to precision

4. When students make assumptions and approximations to simplify a complicated situation they are applying the following mathematical practice:
   ☐ a) Model with mathematics
   ☐ b) Use appropriate tools strategically
   ☐ c) Reason abstractly and quantitatively
   ☐ d) Attend to precision
5. The SMP are:
   a) Primarily for gifted and talented students
   b) Primarily for high school students
   c) Important for all grade levels (K–12)
   d) Both “a” and “b”

6. In the CCSS, the SMP describe what it means to do and use mathematics.
   a) True
   b) False

7. Students are only responsible for mastering the content standards since it is not possible to assess the SMP.
   a) True
   b) False

8. The mathematical practice "Attend to precision" emphasizes using the correct number of decimal places for accuracy in student answers.
   a) True
   b) False

9. The mathematical practice "Modeling" is focused on showing students how to do a procedure.
   a) True
   b) False

10. Students are not expected to apply all eight SMP in every mathematical activity.
    a) True
    b) False