# Kindergarten through Grade Twelve Standards for Mathematical Practice

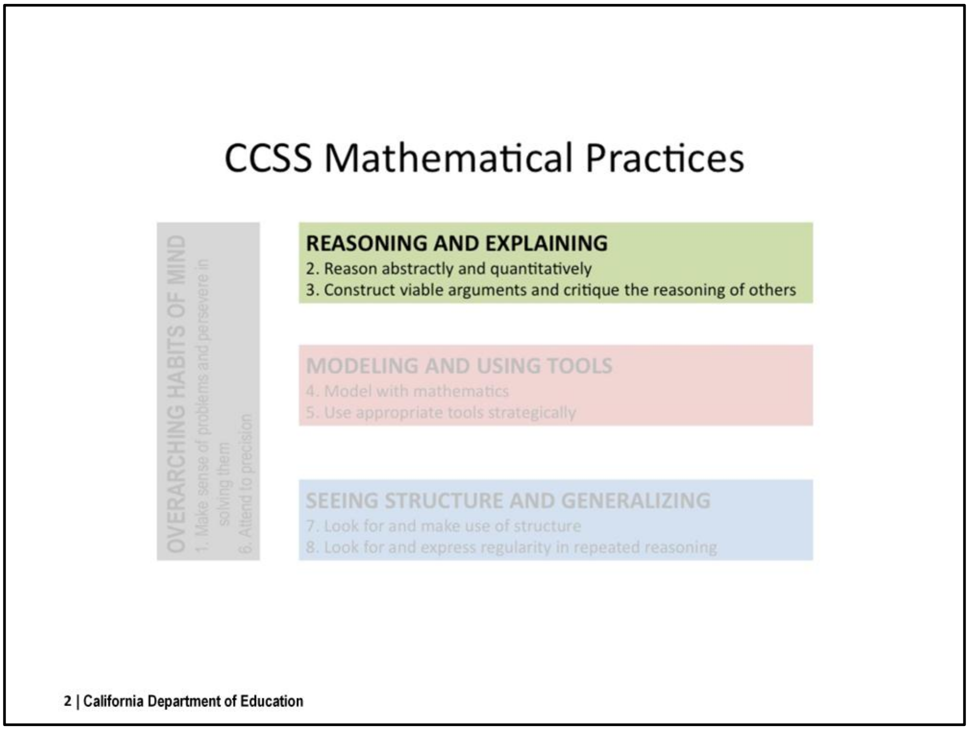
# Unit 3:

# Reasoning and Explaining: MP2 and MP3

CALIFORNIA DEPARTMENT OF EDUCATION
Tom Torlakson, State Superintendent of Public Instruction

**Talking Points:**

Welcome to Unit 3.




**Talking Points:**

- In this unit, we will review the "Reasoning and Explaining" practices; MP2 and MP3.

**Facilitator Note:** Refer participants to the "CCSS Mathematical Practices" handout used in the previous section (**Handout 2.0**).

# Unit 3 Learning Objectives

- You will be able to describe why, to be successful in mathematics, all students need to reason and explain.
- You will be able to explain what it means for students to reason abstractly and quantitatively.
- You will be able to explain what it means for students to construct viable arguments and critique the reasoning of others.

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**Talking Points:**

- Unit 3 examines MP2 and MP3, *Reasoning and Explaining*; specifically how to reason abstractly and quantitatively, construct viable arguments, and critique the reasoning of others.
- By the end of this unit…[review bullets on slide]

# Unit 3 Overview

Focus on how students learn to:

- Construct viable arguments
- Respond to the reasoning of others
- Increase the viability of their arguments

Unit 3 highlights 5th-grade students learning to use the reasoning and explaining practices.

Teachers of all grade levels can transfer concepts learned into their instructional practices.

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**Talking Points:**

- This unit will focus on Focus on how students learn to:
  - Construct viable arguments
  - Respond to the reasoning of others
  - Increase the viability of their arguments
- The unit highlights a 5th-grade classroom as the students learn to use the reasoning and explaining practices. Although the focus is on one grade level, teachers of all grade levels can transfer the concepts learned into their instructional practices.

# Unit 3 Overview, Cont.

- Unpacking MP2 and MP3 and Introduction to 5th Grade Classroom
- Beginning to Reason: Definitions and Conjectures
- Explaining and Justifying
- Identifying Flaws in Reasoning
- Making Arguments More Viable
- Summary and Reflections

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**Facilitator Notes:**

- Review slide.

## Unpacking MP2 and MP3

Read MP2 and MP3

- Highlight key words or phrases that seem particularly cogent to you or that puzzle or intrigue you
- Make a note of questions you have about particular parts of these two mathematical practices.
- Consider in particular how the two practices are related.

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**Talking Points:**

- Take a few minutes to read the reasoning and explaining practices; MP2 and MP3. You will deepen your understanding of the various aspects of each standard as you work through this unit.

[Review bullets on slide]

**Facilitator Notes:**

- Refer participants to the reasoning standards MP2 and MP3 **(Handout 3.0.1)**.

## Small Group Discussion

- What key words or phrases did you highlighted and why they were important to you?
- What questions do you have about these two mathematical practices?
- How are the two standards related?
- What experiences do your students have representing solutions, sharing strategies, listening to other students, and questioning others?
- How can you use this information to build toward MP2 and MP3?

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**Facilitator Notes:**

- Review questions with participants.
- Allow time for discussion in small groups and whole-group sharing upon completion.

## Noticing Students' Thinking

Throughout Unit 3 you will follow the same 5th-grade students as they learn to engage in viable arguments.

*"If discussions about what constitutes a valid justification do not occur, students often rely on the superficial aspects of an argument, such as the use of formal mathematical symbols, over the mathematical reasoning that underlies the argument."*

Healy & Hoyles, 2000

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**Talking Points:**

- Throughout Unit 3 you will be asked to engage in "professional noticing of children's mathematical thinking" (Jacobs, et al., 2010) around the reasoning and explaining practices in a 5th-grade classroom.
- You will be asked to reflect upon the practices illuminated in classroom videos and student work, and to record your observations.

# Introductions

The 5th-grade students featured in this unit participated in a public lesson study (coached by the California Mathematics Project) and focused on the reasoning and explaining practices.

- Students Represented: 44% English learners; 67% free and reduced lunch; 8% special needs

As you watch the video, consider how the students define and value viable arguments:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11837

**What are the descriptors of viable arguments that Jeanette, Haley, and Leslie speak about?**

**How do they describe viable arguments as part of their learning process?**

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**Facilitator Notes:**

- For additional MARS tasks and reengagement lessons visit www.insidemathematics.org.
- Video length: 8 minutes. Prepare Internet video hyperlink: http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11837
- After the video, have participants discuss with partner and then share out with whole group.

**Talking Points:**

- Throughout Unit 3, you will follow the same 5th-grade students as they learn to engage in viable arguments.
- The classroom teacher, Lynn Duri, is a participant in professional learning with the California Mathematics Project, the North Bay Math Project, and the California Math Science Partnership Grant, Project LEAD.
- The professional learning includes: Lesson study, Mathematics Assessment Resource Services (MARS) Tasks (also in Unit 5), and re-engagement lessons.
- The 5th-grade students featured in this unit focus on the reasoning and explaining practices. They received coaching from Harold Asturias, Director of the Center for Mathematics Excellence and Equity (Lawrence Hall of Sciences at the University of California, Berkeley), and Patrick Callahan, co-director of the California Mathematics Project.
- The following video was filmed on the last full day of instruction and demonstrates how students communicated their thinking and investment in continuing their learning.
- As you watch the video, consider how Jeanette, Haley, and Leslie define and value viable arguments:
  - What are the descriptors of viable arguments that Jeanette, Haley, and Leslie speak about?

- How do they describe viable arguments as part of their learning process?

## Connections Across Standards

Compare reasoning practices in the different standards:

- CCSS for Mathematics Standards
- ELA College and Career Readiness (CCR) Anchor Standards
- Partnership for 21st Century Skills

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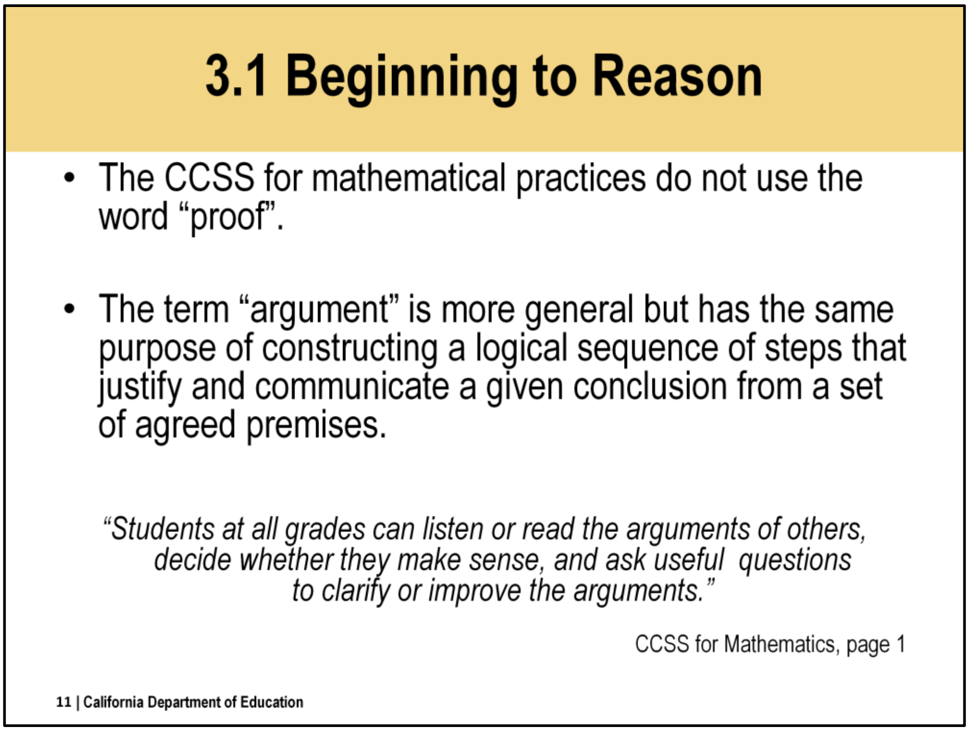
**Talking Points:**

- Compare and contrast the three sets of standards in the “Comparing Standards” handout **(Handout 3.0.2)**
  - What do you notice about the similarities and differences?
  - What practices are valuable for you to emphasize in your classroom?

**Facilitator Notes:**

- Have participants work in small groups to read and discuss the handout.
- Share findings with whole group.




**Talking Points:**

- The CCSS for mathematical practices do not use the word “proof”. This is because many people are familiar with a narrow definition of proof (e.g., a two-column geometry proof).
- The term “argument” is intended to be more general in form but ultimately has the same purpose of constructing a logical sequence of steps that justify and communicate a given conclusion from a set of agreed premises.

## Definitions and Conjectures

- When students begin to reason, they need to establish definitions (e.g., clear and logical descriptions of *what* they are reasoning about).
- They also need to make conjectures — informed guesses of what is true — they then set out to explain through reasoning.

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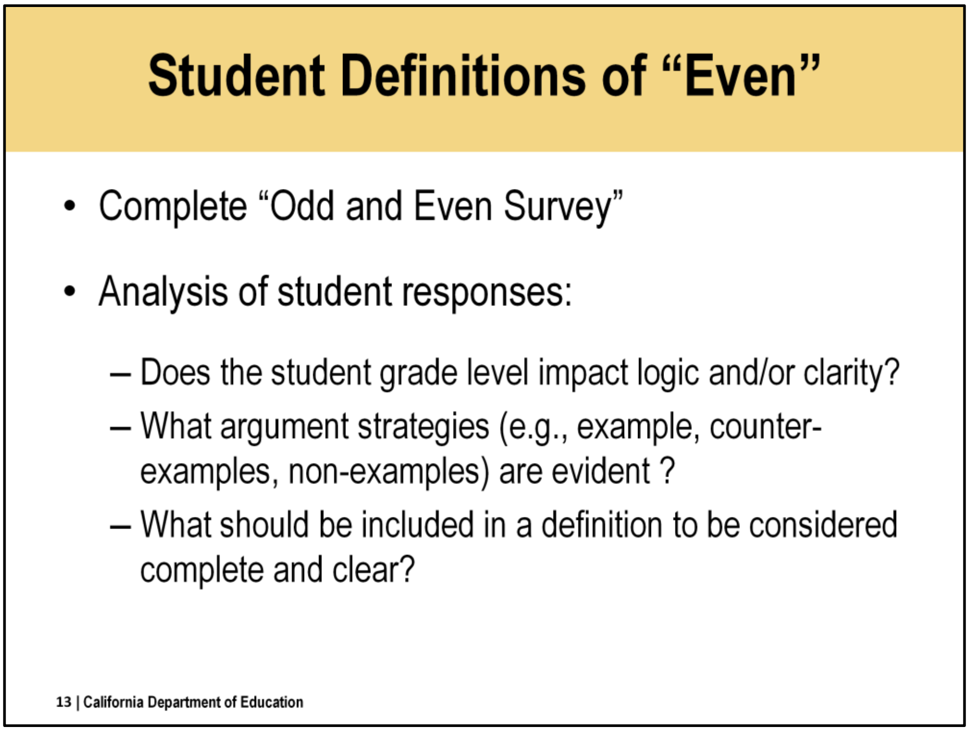
**Facilitator Notes:**

Review bullets on slide.

**Talking Points:**

We will now learn how a 5th grade classroom lesson inducted students into reasoning.

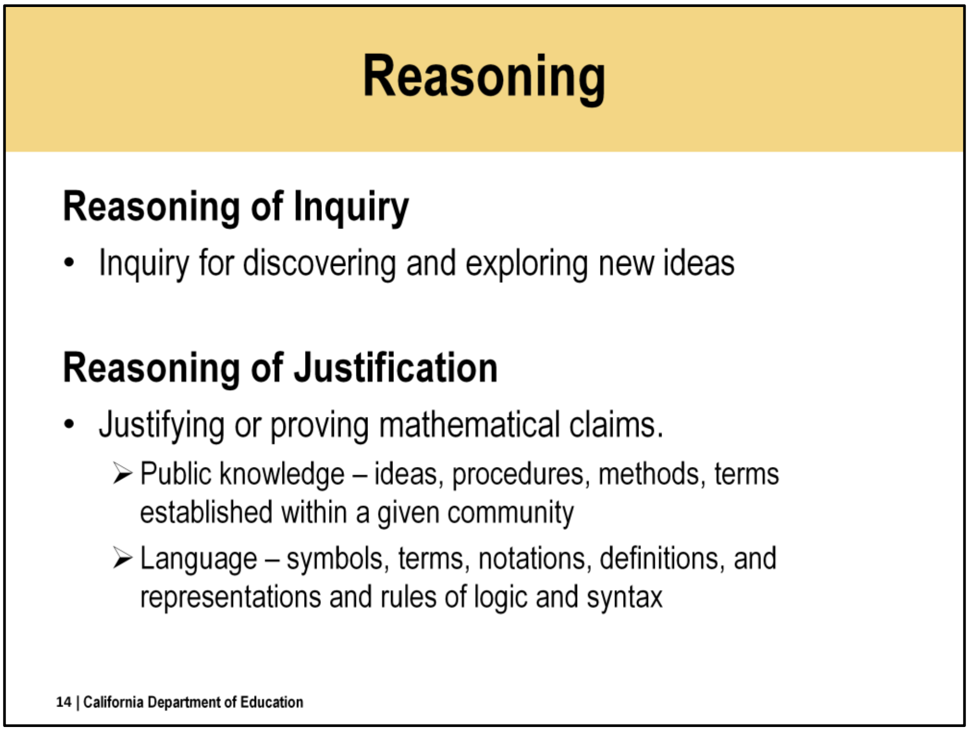



**Facilitator Notes:**

- Refer participants to handout: “Odd and Even Survey” **(Handout 3.1.1)**.
- After participants complete their survey and share some responses, refer them to the sample of student responses, “Student Definitions of Even” **(Handout 3.1.2)**.

**Talking Points:**

- Over 300 students from grade five through high school geometry responded to the survey. Consider the clarity and logic of the student-generated definitions. Does the students’ grade level impact the logic and/or clarity?
- In small groups, discuss what argument strategies (e.g., example, counter-examples, non-examples) are evident in the student definitions. What should be included in a definition to be considered complete and clear?




**Talking Points:**

- Mathematical reasoning can serve as an instrument of inquiry for discovering and exploring new ideas, a process that we call the *reasoning of inquiry*.
- Mathematical reasoning also functions centrally in justifying or proving mathematical claims, a process that we call the *reasoning of justification*.

- The reasoning of justification, as we see it, rests on two foundations. One foundation is an evolving body of public knowledge — the mathematical ideas, procedures, methods, and terms that have already been defined and established within a given community.
- The second foundation of mathematical reasoning is mathematical language — symbols, terms, notations, definitions, and representations — and rules of logic and syntax for their meaningful use in formulating claims and the networks of relationships used to justify them.

## Developing a Community of Reasoners

**Three domains of work (Ball, 2002):**

- Select tasks and provide opportunities for reasoning
- Make knowledge public and scaffold the use of mathematical language and knowledge
  - Make mathematical records (through notebooks, public postings) to make the work public and available for public development
- Develop a classroom culture in which students
  - Develop serious interest in and respect for others' ideas
  - Attend and respond to, as well as use, others' solutions or proposals as a means of strengthening understanding

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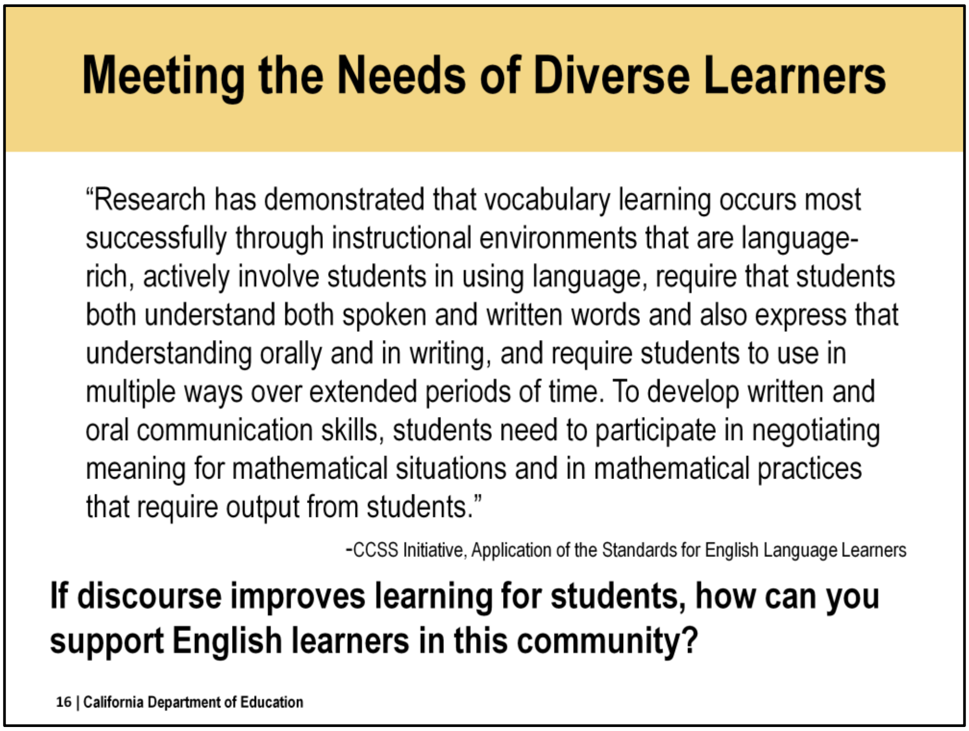
**Talking Points:**

- Developing a reasoning community involves three domains of work a teacher must do:
  - Selecting mathematical tasks that need and provide opportunities for mathematical reasoning.
  - Making mathematical knowledge public and scaffolding the use of mathematical language and knowledge: Making mathematical records (through notebooks, public postings) to make the work public and available for public development.
  - Establishing a classroom culture permeated with serious interest in and respect for others' mathematical ideas. Students learn to attend and respond to, as well as use others' solutions or proposals as a means of strengthening their own understanding and the subsequent contributions they can make to the class work.

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**Notes to Facilitator:**

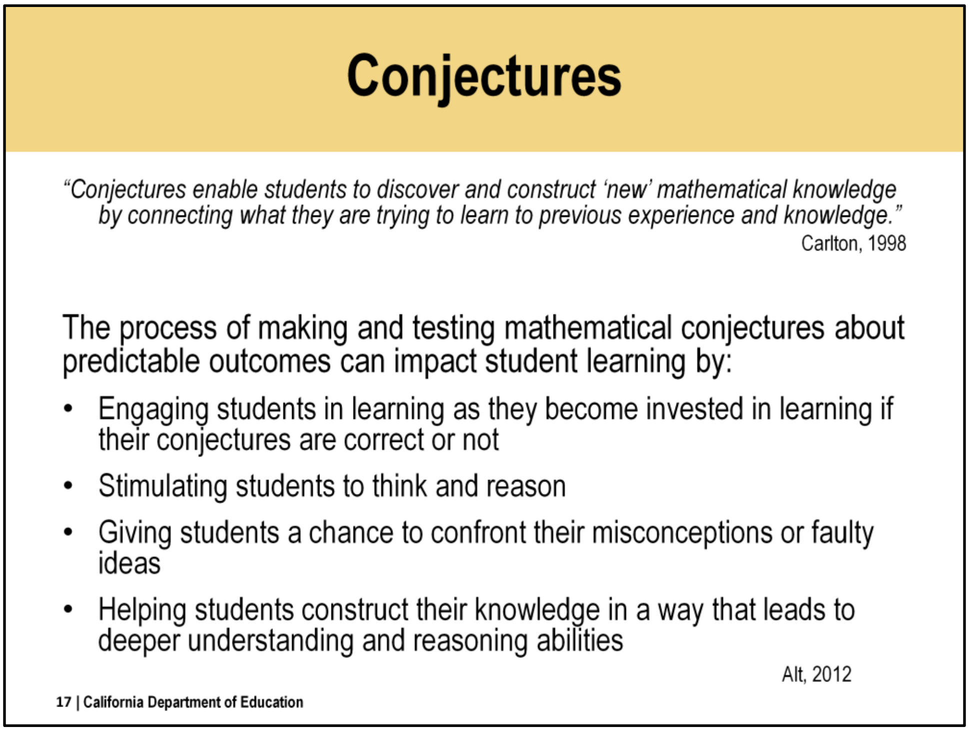
- Refer participants to the handout, “Sample Strategies for Differentiating Instruction” (**Handout 3.1.3)** and have them read silently.

**Facilitator Notes:**

Turn attention to slide an allow participants to read slide silently.

**Talking Points:**

- Ball's description of a community of reasoners can be integrated with strategies for teaching English learners, students with special needs, and high-achieving students.
- Refer to “Sample Strategies for Differentiating Instruction” (**Handout 3.1.3)** for examples of strategies to meet the needs of diverse learners in your classroom. [Allow time to read and support discussion as time allows]
- Discuss in small groups: If discourse improves learning for students, how can you support English learners in this community?




**Talking Points:**

- Why Are Conjectures Valuable for Students?
- The process of students making mathematical conjectures about predictable outcomes — and then testing these conjectures — can impact student learning in the following ways:

[Review bullets on slide]

## Introducing Viable Arguments

Harold: What do you know about odd and even?

Leslie: Odd is something that doesn't, like you can't have it equal. Number 7 is 4 and 3, not 4 and 4, like 8.

Max: Odd numbers if you add it, you'd have to add two

different numbers, not like the same; and if you divide it, you get 1 remainder.

Daniela: Odd numbers don't have partners, and even numbers do have patterns. Four, there are two fingers for each. Three has one left over. It doesn't have a partner, so it's not even.

The following video captures Daniela's pairing of fingers:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11843

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**Notes to Facilitator:**

- Refer participants to the "Lesson Plan for Day One" handout **(Handout 3.1.4).**
- Movie length: 20 seconds. Prepare Internet video hyperlink: http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11843

**Talking Points:**

- The fifth grade work began with a lesson taught by guest teacher, Harold Asturias.
- The lesson opened with Asturias stating, "mistakes are gifts because they promote discussion," "ask questions until it makes sense," and "think with language and use language to think." The lesson then tapped into student prior knowledge.
- The slide shows three student definitions [read slide]
- The following video captures Daniela's pairing of fingers as a representation of the concept of even and odd numbers.

Video available on the Brokers of Expertise Web site (SMP Module section 3.1.3) at http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11843

## Misconception: “50 is both odd and even”

**Instructional conjecture:** If students view a video of another student’s misconception of odd and even, they would be motivated to develop an argument to help this student clarify her understanding. In turn, the class would develop a common definition that would become part of the classroom’s public knowledge.

A video of Emily was filmed immediately after she successfully participated in a textbook lesson defining even numbers as ending in 0, 2, 4, 6, 8:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11844

**Identify the cases that Emily shared.**

**What is her misconception?**

**How would you help her?**

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**Notes to Facilitator:**

- Movie length: 2 minutes. Prepare Internet video hyperlink: http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11844

**Talking Points:**

- The lesson presented students with a “significant problem designed to engage them … and elicit conjectures” (Clements, 2004).
- Building on Deborah Ball and Deborah Shifter’s odd and even work with students, the instructional conjecture was that if students viewed a video of another fifth grade student’s misconception of odd and even, students would be motivated to develop an argument to help this student clarify her understanding. In turn, the class would develop a common definition that would become part of the classroom’s public knowledge.
- A video of Emily was filmed immediately after she successfully participated in a textbook lesson defining even numbers as numbers ending in 0, 2, 4, 6, 8. In the video, she demonstrated that although she was successful in the textbook lesson about even numbers, she continued to hold on to a misunderstanding about what “even” means.
- Let’s watch the video in which Emily’s cases/examples seemed “reasonable” to the fifth grade audience, causing her classmates to question their understanding of odd and even numbers.

[after viewing video]

- The movie caused greater disequilibrium among students than was anticipated. Emily’s flawed reasoning greatly influenced the students and their fragile understanding of odd and even numbers became evident. This discovery enabled the teacher to adjust the content focus for teaching reasoning practices during the

school year.

## Forming Conjectures

Daniela's explanation:

"We figured out that 50 is an even number. And how we found that out is … because we first wanted to prove that (grouping) in any way, meaning in threes, fours, fives, or tens … we wanted to prove that it is even. So we tried (dividing into groups of) three, and it worked. But we had two left over, and we figured out that that's still even, because the two has a partner. So it was even in any way, even if you have leftovers, the leftovers have even blocks."

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**Talking Points:**

- Daniela has a beginning conjecture that a number can be divided by any divisor, and if there is a remainder of 2, the number will be even, because the remainder is a pair and pairs are even.
- Notice her use of multiple definitions of the word "even" in the transcript above.

## Achieving Consensus

*"Evidence shows that class discussion is important in students' development of mathematical conceptions … instances of disagreement arise from diverse ideas generated by children."*

Wood, 1999

As the work progressed, the students developed two conjectures the class agreed to:

- If you put cubes into pairs, and there are no leftovers, then the number is even.
- If you put the cubes into pairs, and there is a leftover, then the number is odd.

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**Talking Points:**

- The students developed two conjectures the class agreed to: [refer to slide]

# Other Conjectures

- Even means the "same amount." If you divide a number by any divisor, and it doesn't have a leftover, then it's even.
- If you divide a number by some divisor and there is no leftover, then the number is even. If you divide the same number by a different divisor, and there is a leftover, then that shows that the number is odd. For example:

15 ÷ 3 = 5 (even)
15 ÷ 2 = 7 R-1 (odd)

... so 15 is both odd and even …


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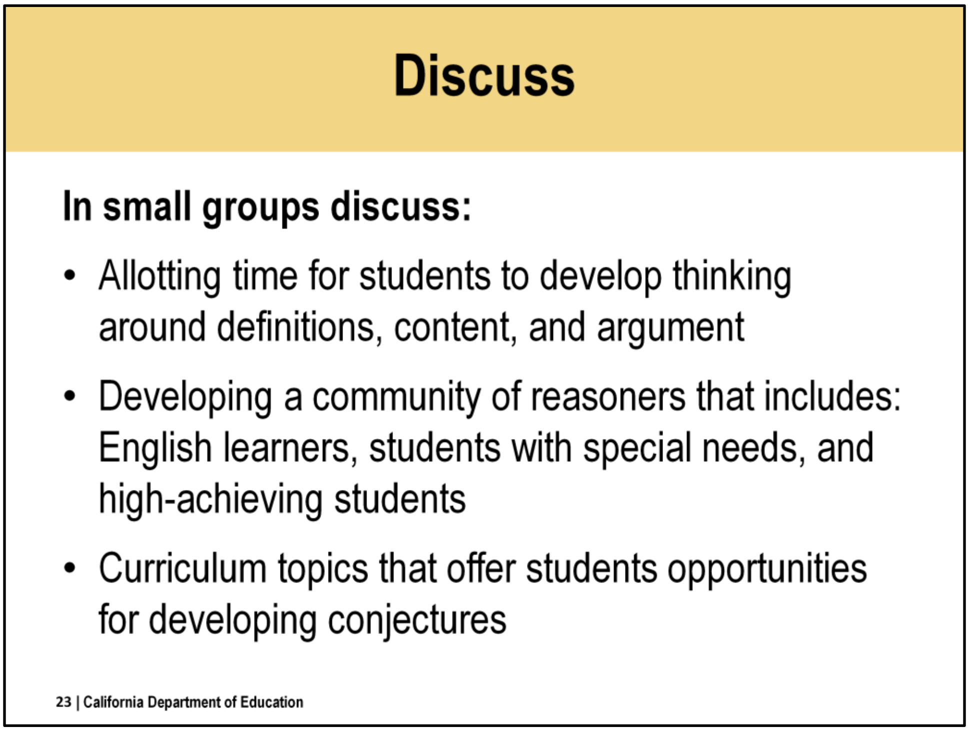
**Talking Points:**

- These two conjectures were brought forward without class consensus and formed the basis for ongoing arguments.




**Facilitator Notes:**

Allow time to discuss the three topics on the slide.

## 3.2 Explaining and Justifying

*"Higher order questions generally challenge the student to provide additional information and engage in deeper understanding and reflection, and ultimately promote greater conceptual development ."*

Nathan & Kim, 2007

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- Explaining and justifying are at the heart of mathematical reasoning. By becoming aware of the language patterns of students, teachers are able to scaffold mathematics instruction and support students in moving toward justifications and generalizations.
- Two studies that focused on discourse (Mendez, 2004) and teacher elicitation (Nathan & Kim, 2007) found that explanations, justifications, and generalizations coincide with the reasoning practices of students. This work provides the basis for a hierarchy of questions and responses between teachers and students. The researchers found that teachers move up and down the hierarchy of questioning in response to student needs.

## Taxonomy of Questions in Mathematical Discourse

When students make their thinking visible, they are organizing their thoughts, connecting to previous understandings, and recognizing and correcting gaps in their logic.

- Review the “Taxonomy of Questions in Mathematical Discourse” handout.

An awareness of language patterns enables the teacher and other students to access student thinking which helps to develop a public knowledge base. This awareness is especially helpful in supporting English learners and students with special needs.

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**Talking Points:**

- When students make their thinking visible, they are organizing their thoughts, connecting to previous understandings, and recognizing and correcting gaps in their logic.
- An awareness of language patterns enables the teacher and other students to access student thinking which helps to develop a public knowledge base. This awareness is especially helpful in supporting English learners and students with special needs.

**Facilitator Notes:**

- Refer participants to handout, “Taxonomy of Questions in Mathematical Discourse: Questions and Responses” **(Handout 3.2.1)**.
- Allow participants to read and reflect.
- Review question types and language of questions and responses

# Levels of Explanations

Level 1: Abdicating reasoning to an external authority

Level 2: Systematically working through multiple explanations

Level 3: Beginning to notice patterns or trends

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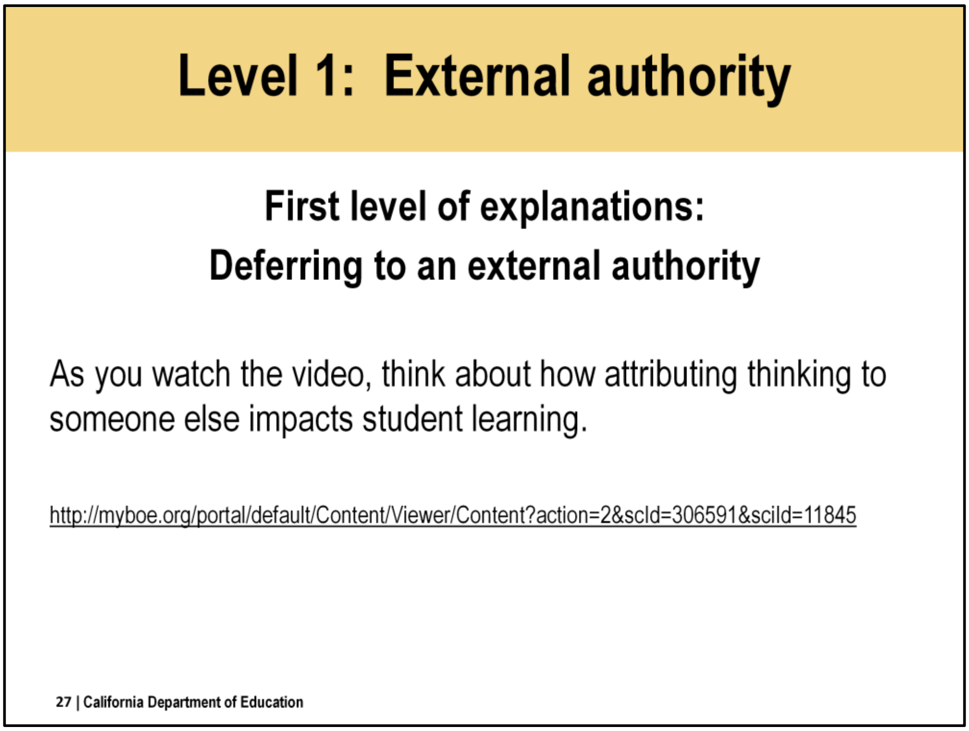
**Talking Points:**

- The ability to identify patterns is what allows students to ponder if something will always work and make generalizations.
- As you observe the following videos, student work, and transcripts consider how to successfully move students through the levels of explanation.




**Notes to Facilitator:**

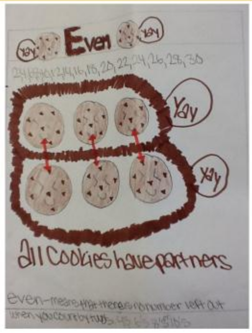
- Movie length: 30 seconds. Prepare Internet video hyperlink:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11845

**Talking Points:**

- The interview on this short video clip reveals an interesting take on the first level of explanations; deferring to an external authority.
- As you watch the video, think about how attributing thinking to someone else impacts student learning.
- Lorena challenges her parents who are deferring to an external authority — "I know because the teacher told me."

# Level 1: Perceptual Proof





- Above are single examples demonstrating what an odd number is.
- In the video, students are satisfied with one case as sufficient evidence to validate their claim. This is referred to as a "perceptual proof."

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11846

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**Facilitator Notes:**

- Movie length: 1 minute. Prepare Internet video hyperlink:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11846

**Talking Points:**

- *"When learning to explain their thinking, students will provide one example and believe it is enough evidence to argue their conjecture. This is referred to as a perceptual proof."* (Sowder & Harel, 1998)
- In the video, Jason and Jonathan use a single example to demonstrate what an odd number is.
- In the two examples (posters and video) presented, students are satisfied with one case as sufficient evidence to validate their claim.

Student writing on posters:

- Even. Yay 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30. All cookies have partners. Even means there are no numbers left out when you count by twos.
- Boo! Odd, Boo! Not all ghosts have partners. 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29. Odd means there is a number left out when you count by 1s, 3s, 5s, 7s, or 9s.

## Level 2: Multiple Examples

**Second level of explanations:**
**Learning to present an argument by presenting multiple examples or cases**

- Demonstrates that a claim works in more than one instance.
- Students often get into an argument about how much evidence is enough, especially when they realize they cannot test every case.
- In the video, Brandon, Chris, and Nathan use multiple examples organized systematically:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11847

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**Facilitator Notes:**

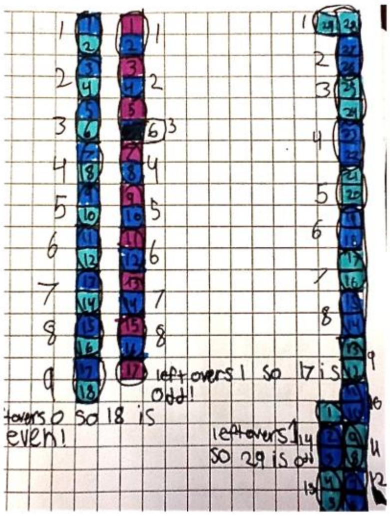
- Movie length: 2 minutes. Prepare Internet video hyperlink:

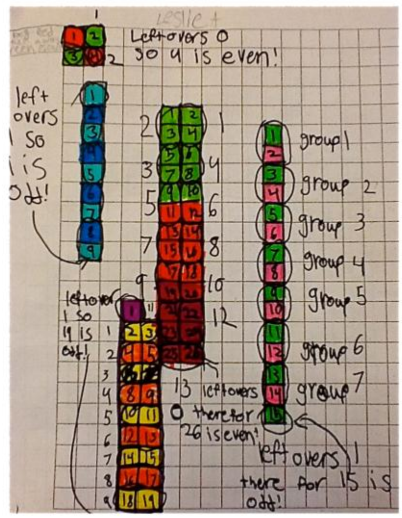
http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11847

**Talking Points:**

- The second level of explanations is learning to present an argument by presenting multiple examples or cases demonstrating that a claim works in more than one instance.
- Students often get into an argument about how much evidence is enough, especially when they realize they cannot test every case.
- In the video, Brandon, Chris, and Nathan use multiple examples organized systematically.

## Level 2: Multiple Examples



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**Talking Points:**

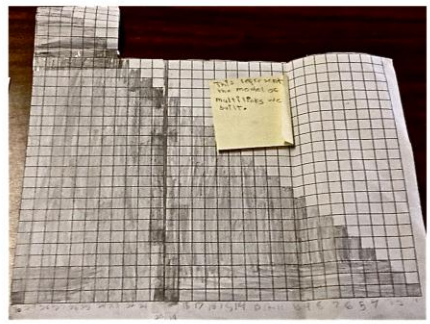
- Let’s take a look at the photos of Leslie’s tests for odd and even numbers.
- For the test, Leslie chose random numbers between 1 and 30 (9, 15, 17, 18, 26, and 29).
- She defined “even” as a number where there is no remainder, so every cube has a partner.
- She began by building trains with multi-link cubes using colors to show groups of two.
- She replicates the trains on graph paper using the same color-coding, labels each cube with a number, and labels the number of groups of two.
- She writes for each train: Leftover: ___, so /“there for” ___ is odd/even.

# Level 3: Noticing Patterns

**Third level of explanations:**
**Multiple representations with emerging pattern(s)**

Students begin to see patterns across multiple representations.

- Being able to identify and understand these patterns supports learners as they begin to abstract and generalize.
- Review Handout 3.2.2, “Level 3: Noticing Patterns Across Multiple Examples”



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**Talking Points:**

- In the third level of explanations, students begin to see patterns across multiple representations.
- Being able to identify and understand these patterns supports learners as they begin to abstract and generalize; in this case, finding the method to decide whether any number is odd or even.
- As you saw in the previous video, Brandon, Chris and Nathan built their cube trains for numbers 1 through 30 and have made a representation of their trains (see photo on slide). The Post-It® reads: “This represents the model we built with multi-links.”

**Facilitator Notes:**

- Refer participants “Level 3: Noticing Patterns Across Multiple Examples” **(Handout 3.2.2)** for dialog of the boy’s thinking.

# Level 3: Patterns Across Examples

Watch Brandon, Chris, and Nathan working across strategies:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11848

Discuss in small groups:

- The evidence students use in their explanation to argue for their claim.
- Does the argument include elements of an accurate generalization?
- How will this impact their understanding of this concept?
- How does the work with explanations presented in this unit intersect with the recommendations for English learners? Students with special needs? (Refer to Handout 3.2.3 "Engaging Diverse Learners")

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**Facilitator Notes:**

- Movie length: 43 seconds. Prepare Internet video hyperlink: http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11848

**Talking Points:**

- This video shows Brandon, Chris, and Nathan working across strategies. [show video]
- Take a few minutes to discuss the questions on the slide in small groups.
- Refer to "Engaging Diverse Learners" **(Handout 3.2.3)** for the last question.

[share out with whole group as time allows]

## 3.3 Flaws in Reasoning

*"Argument is central to thought and the construction of knowledge (e.g., Kuhn, 1992). The significance of argument to conceptual understanding in mathematics is related to the development of students' thinking and reasoning that occurs during the acts of challenge and justification."*

Wood, 1999

*"Students learn about mathematics by exploring their own and others' reasoning in problem-solving situations. Exploring the reasoning of self and others allows for flaws in thinking to be revealed and corrected. Opportunities for students to reveal their thinking and for their peers to evaluate and contribute to the improvement of student thinking can lead to stronger mathematical understanding. As students become more mathematically proficient and their reasoning skills increase they should be able to identify flaws in their own and others' thinking; thus prompting revision of thinking that leads to better problem solving."*

Daro, 2012



**Facilitator Notes:**

- Review quotes on slide.
- Allow for discussion if time allows.

## Common Flaws

Several common flaws emerged as the 5th-grade students made their arguments part of the public conversation.

Competing definitions

- A number that can be divided evenly by any divisor without a remainder
- A number that is a multiple of two

Pronoun referents

"Odd numbers if you add it, you'd have to add two different numbers, not like the same; and if you divide it, you get one remainder."

Partial understanding or explanations

- "Odd means there is a number left out when you count by 1s, 3s, 5s, 7s, or 9s."
- "I think two decides, because two is always even."

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**Talking Points:**

- Several common flaws emerged as the 5th-grade students made their arguments part of the public conversation.
  - The belief that "even" means divided evenly (without a remainder) is invalidated by the equation 15 ÷ 3 = 5. Furthermore, if both definitions of "even" are used, then a number can be both odd and even (e.g., 15 ÷ 3 = 5 [even] and 15 ÷ 2 = 7 R-1 [odd]).
  - In some student explanations, there is a lack of clarity as to what a pronoun refers to.
  - Some students demonstrated partial understanding, such as: "Odd means there is a number left out when you count by 1s, 3s, 5s, 7s, or 9s" and "I think 2 decides, because 2 is always even."

- This $5^{th}$-grade community is novice in its reasoning, and as the students learn to handle their flaws for odd and even, more sophisticated reasoning skills can become the focus of instruction.

# Identifying Flaws through Discourse

## Discussion: Refer to Handout 3.3.1

**Community of Reasoners:**

- Who is leading the conversation?
- Do you see a respect for others' mathematical ideas? What is the evidence?
- How often does the teacher intervene and for what purpose(s)?
- Where do you see evidence of students referring to, building on, and/or challenging other's thinking?

**Arguments:**

- What questions do the students pose?
- What reasoning does Chris use? Max?
- What flaws are students identifying?

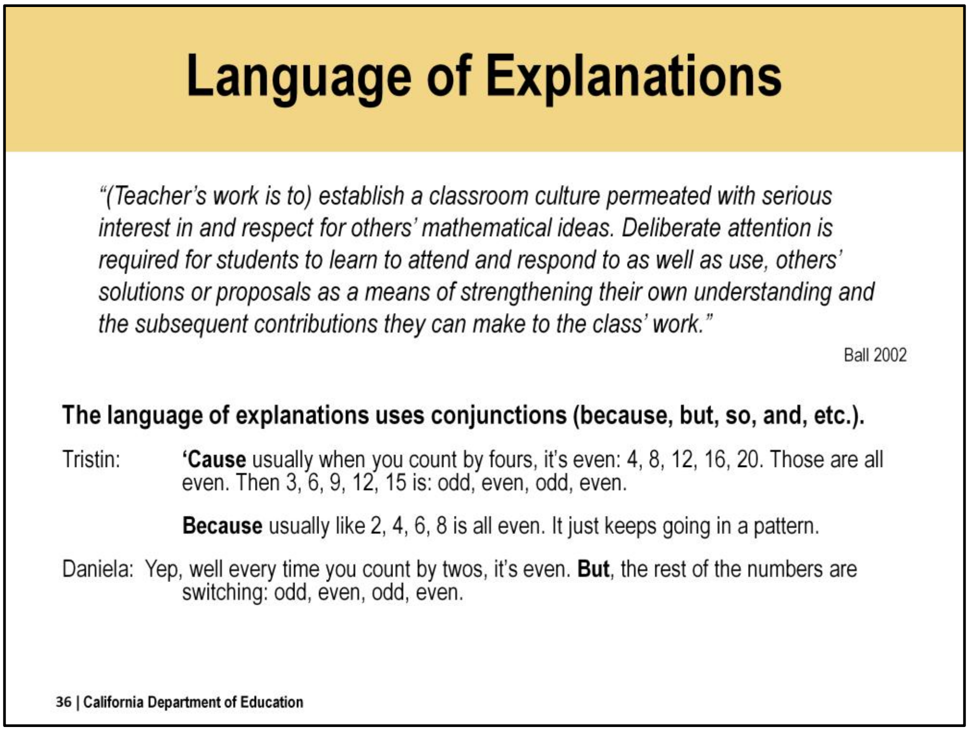
**Flaws:**

- Does language cause confusion?
- Are there partial arguments?
- What is missing?



**Facilitator Notes:**

- Refer to the dialogue on "Identifying Flaws Through Discourse" **(Handout #3.3.1).**
- Provide time for partner or small group discussion of questions on slide.




**Talking Points:**

- In previous work analyzing mathematical language (see section 3.2) patterns of language become apparent. As you learn to listen for and recognize these patterns, they can be modeled by "revoicing" (Chapin, et al., 2012) student language.
- You can provide "just in time" language support to students as they are explaining their thinking. This enables us to "support English learners as they engage in complex mathematical language" (Moschkovich, 2012).

# Language of Justification

**The language of justification uses an "If…. then… because…" structure. In many cases, "then" is implied.**

Max: I think what it's trying to say is that **if** it's 3, 3, and 3, (**then**) those are all equal numbers, **but** 3 is odd.

**Justifications often include modals (would, should, could, might have, must). This language may need some additional support.**

Chris: **If** you count by twos, (**then)** you **would** just get an even number, and you'll never land on an odd.
But like Tristin said, "…it goes in a pattern: odd, even, odd, even. And, **if** one number was both, **(then**) it **would** break the pattern. I don't think a number can be both even and odd."



**Talking Points:**

- The language of justification uses an "If…. then… because…" structure. In many cases, "then" is implied. [review Max example)
- Justifications often include modals (would, should, could, might have, must). This language may need some additional support.
- Teachers may respond to these types of student language patterns by:
  - Scaffolding student responses. A teacher may listen to students present their explanations and add "because…", thus prompting students to add to their response.
  - Using sentence frames such as "If …. then…. because….". This frame communicates the parts of the response that are needed and the language structure, but does not limit student thinking.
  - Moving up and down the questioning hierarchy helping students to fill in gaps in knowledge and moving the class to the levels of discourse where students are developing meaning.

## Flaws in Reasoning

A persistent argument among the students was that a number could be both odd and even. Watch three students explain the flaw in the others' reasoning:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11849

Watch as three groups of students in the classroom demonstrated misunderstandings when asked to define what a mathematical "rule" was:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11850

Teachers may assume that students understand that a mathematical rule is a statement that is always true. The definition exercise demonstrated that in the daily activities of students, rules often have exceptions.

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**Facilitator Notes:**

- Movie length (first video): 2 minutes 36 seconds. Prepare Internet video hyperlink: http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11849
- Movie length (second video): 1 minute. Prepare Internet video hyperlink: http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11850

**Talking Points:**

- One of the persistent arguments among the students was that a number could be both odd and even.
- In the first video clip, Brandon, Christopher, and Nathan explain the flaw in the others' reasoning. [play video]
- In the second example, three groups of students in the classroom demonstrated misunderstandings when asked to define what a mathematical "rule" was.
- Their confusion was influenced by everyday experiences with school rules and expectations. For example, one group explained that an example of a rule was that you couldn't play tag on the cement because it is easy to get hurt. However, they explained, there are exceptions made to the rule for physical education classes.
- In the second video, Leslie similarly explains exceptions to rules. The notion of exception allowed students to see 9 as a multiple of 2 as an exception to the rule of even. [play video]
- Teachers may assume that students understand that a mathematical rule is a statement that is always true. The definition exercise demonstrated that in the daily activities of students, rules often have exceptions.

- With a partner, discuss how to help students understand what a mathematical rule is.

## 3.4 Making Arguments More Viable

*"As students become more mathematically proficient and their reasoning skills increase they should be able to identify flaws in their own and others' thinking: thus prompting revision of thinking that leads to better problem solving."*

Daro, 2012

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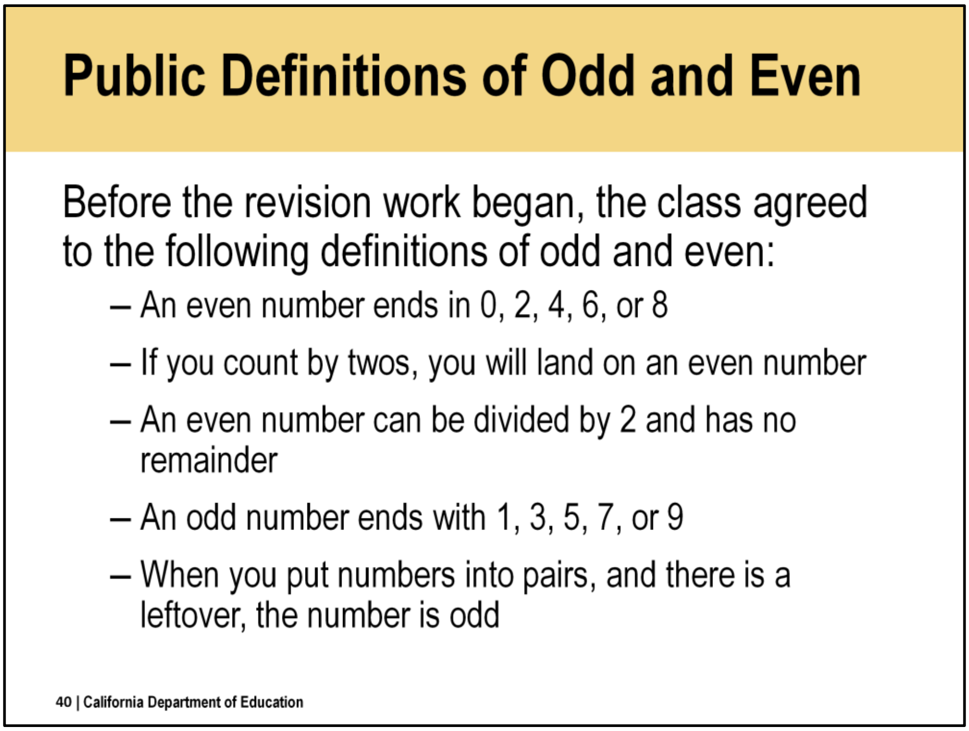
**Talking Points:**

- During the writing process, an author can expect to make several revisions in order to develop a thoughtful, coherent piece.
- Often an editor reads the piece and offers suggestions and an outside perspective.
- The same expectation should be held as young mathematicians revise and improve their reasoning while developing a coherent argument.




**Talking Points:**

- During the writing process, an author can expect to make several revisions in order to develop a thoughtful, coherent piece. Often an editor reads the piece and offers suggestions and an outside perspective. The same expectation should be held of young mathematicians as they revise and improve their reasoning while developing a coherent argument.
- Duri, the 5th grade teacher, offered her students multiple opportunities to share their reasoning with teachers, partners, visiting educators, small groups, and the entire class. Over a period of two weeks, students had multiple opportunities to rehearse and refine their arguments with advice from these outside perspectives.
- Before the revision work began, the class agreed to the following definitions of odd and even: [refer to slide]

## Revision Sequence

Step 1: Complete the MARS Flower Arrangements task

Step 2: Meet individually with the teacher, then revise work

Step 3: Share solution with a partner and revise work

Step 4: Share solution with whole class and revise

Step 5: Work with small group to combine arguments into a poster

Step 6: Share rough draft of poster with another group and revise

Step 7: Share completed poster at public research lesson



**Talking Points:**

- Their sequence of work contained the following steps: [refer to slide]


**Talking Points:**

- The 5th graders worked on the MARS task titled, "*Flower Arrangements*" **(Handout 3.4.1)**.
- Some students needed clarification of what a flower arrangement was, so their teacher used roses, tulips, lilies, and vases to teach the names of the flowers and demonstrate how to choose flowers to be added to an arrangement.

**Facilitator Notes:**

- Refer participants to "Flower Arrangements" **(Handout 3.4.1)**.

## Flower Arrangements: Discussion

Step 1: Complete the “Flower Arrangements” MARS task (Handout 3.4.1).

- Consider how an English learner, GATE, and special needs student might respond to the task.

Discuss:

- Constraints of the problem (e.g., there are more roses than lilies). Note: Students used “clue” as a synonym for “constraint”.
- Questions you would ask students if they skipped any one of these constraints.

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**Talking Points:**

- The first step of the revision sequence was for each student to complete the task.
- [to participants] Individually, please complete the “Flower Arrangements” MARS task provided in your participant packet.
- As you work, consider how an EL student, GATE student, and a special needs student might respond to the task.

**Facilitator Note:**

- After completion of task, allow for discussion of topics on slide.
- Share out as time allows.

## Flower Arrangements: Student Work Sample 1

Chris:

"I did addition and then I just guessed a random number for tulips; then I remembered that there is more tulips than roses and roses than lilies, so I went down by 2 number and then added 5 + 3 + 1 and it equaled 9. I was thinking of an even number being a problem but then in the direction it said, 'She does odd numbers'."

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**Talking Points:**

- The next three slides show student responses to the Flower Arrangements task.
- After reviewing your list of constraints, identify which are addressed and which are missing in this student response.
- **Discuss:** In this case, what questions would you ask to help the student revise his/her written argument?

## Flower Arrangements: Student Work Sample 2

Nathan:

"I figured it out with the clues. Tim's grandmother always uses an odd number, and there are more tulips than roses and more roses than lilies. She is using 9 flowers today which is an odd number [II II II II I]. 5 is odd [II II I], because there is one left out. 3 is odd, because there is one left out [II I]; and 1 is odd, because it has no partner."

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**Talking Points:**

- Identify which constraints are addressed and which are missing in Nathan's response.
- **Discuss:** In this case, what questions would you ask to help him revise his written argument?

## Flower Arrangements: Student Work Sample 3

Spencer:

"I figured this out by looking at the odd numbers and seeing which one I could use without going too high. So I chose numbers 5, 3, 1, and they add up to 9. You can't get to an odd number by going by 2."

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**Talking Points:**

- Identify which constraints are addressed and which are missing in this response.
- **Discuss:** What questions would you ask Spencer to help him revise his written argument?

# Quick Conference with Teacher

Step 2: Students individually meet with teacher, then revise work

Teacher prompts:

- How do you know that 5 is an odd number?
- How did you know that it was 5 and not 7?
- Can you show me another way?
- Why doesn't 4 + 3 + 2 work? It totals 9…
- What does 5 refer to? The 3? The 1?
- Could you use what you learned in the first problem to help with the second problem?

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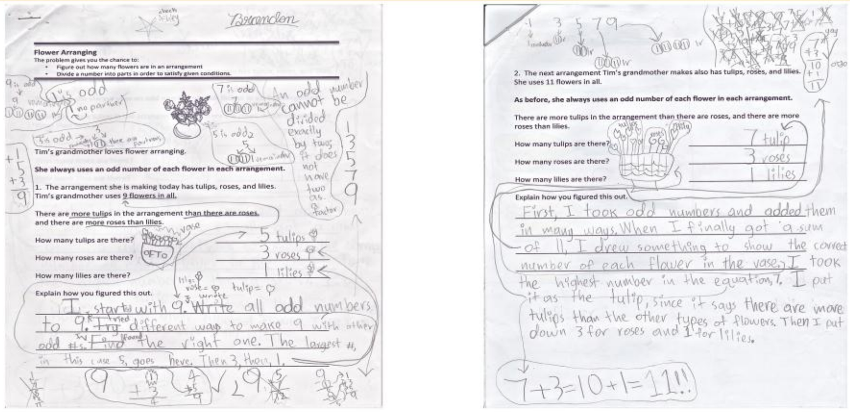
**Talking Points:**

- As a second step, the teacher met with each student, read their responses, and asked questions to prompt more complete responses. The questions asked were: [refer to slide]

# Brandon's Revisions




Does Brandon meet all the constraints of the problem?
How do the additions to his work impact his response? (arrows point to revisions)
Is anything missing?
What would you ask him to consider about his run-on equation: 7 + 3 = 10 + 1 = 11?

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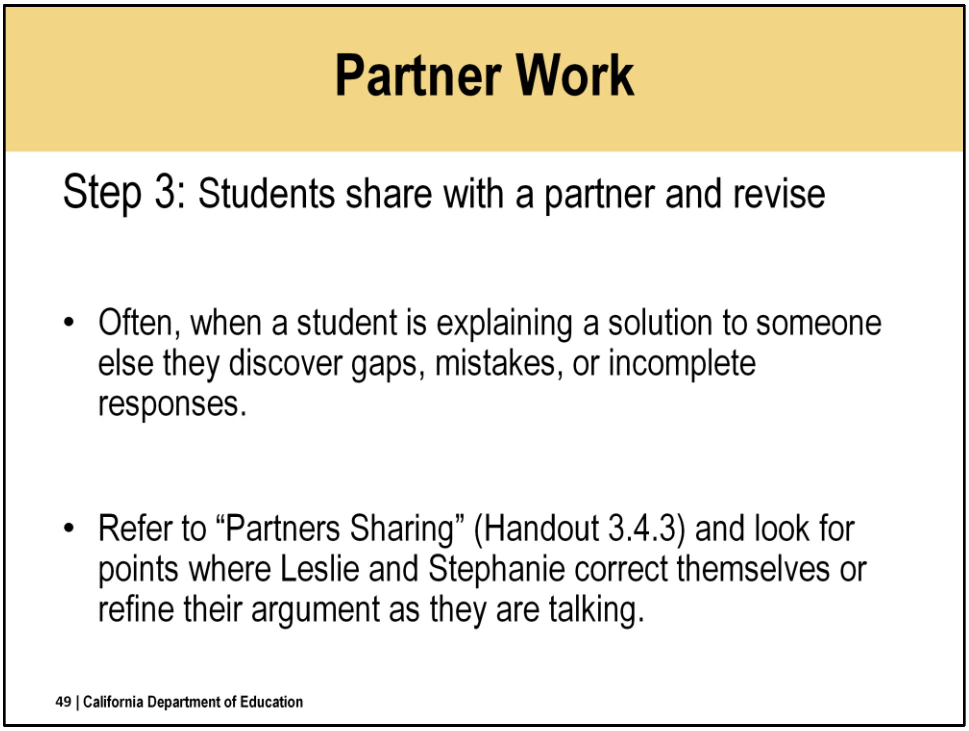
**Facilitator Notes:**

- Refer participants to Brandon's Flower Arrangement (**Handout 3.4.2**)

**Talking Points:**

- Does Brandon meet all the constraints of the problem?
- How do the additions to his work impact his response? (arrows point to revisions)
- Is anything missing? What would you ask him to consider about his run-on equation: 7 + 3 = 10 + 1 = 11?




**Facilitator Notes:**

- Refer participants to "Partners Sharing" **(Handout 3.4.3)**

**Talking Points:**

- Often, when a student is explaining a solution to someone else they discover gaps, mistakes, or incomplete responses.
- In the following dialogue, Leslie has taught her strategy to Stephanie and they share their method with the teacher.
- Look for points where the girls correct themselves or refine their argument as they are talking.

# Using Objects to Share Solutions

## Step 4: Students share work with the whole class then revise

Explaining individual solutions allows for rehearsing an argument and opening it up to questioning.

Public sharing also gives the students an opportunity to view multiple representations and strategies.

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**Talking Points:**

- Students were asked to explain to the class their individual solutions. This experience allows for rehearsing their argument and opening it up to questioning.
- The public sharing also gives the students an opportunity to view multiple representations and strategies, and rehearse the class definitions of odd and even.
- This activity prepared students for a follow-up activity in which they would make a group poster that included several strategies, representations, and public definitions.

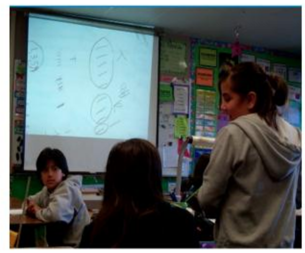
**Lorena’s Solution**

- Lorena knew the total was 9 flowers. She knew that 3 + 3 + 3 = 9, so she arranged three groups of 3 pens. 3 is an odd number.
- She knew there were more tulips than roses, and more roses than lilies, so there could not be the same number of each flower.
- She took 2 pens from one group of 3 and added them to another group of 3, which made 5 tulips + 3 roses + 1 lily = 9 flowers in the arrangement.

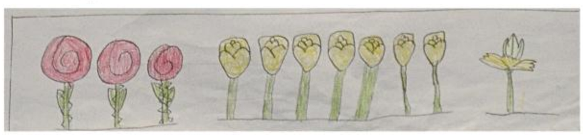
# Using Objects:
## Finger Method, Tally Marks, and Drawings

Daniela uses her fingers as a tool. She begins by putting out nine fingers and then adjusts the numbers of fingers to meet the constraints of the problem.

Some students used tally marks to decontextualize the problem.



Other students decontextualized the problem using lists of odd numbers and cubes, and then used drawings of flowers to re-contextualize their solution.





**Facilitator Notes:**

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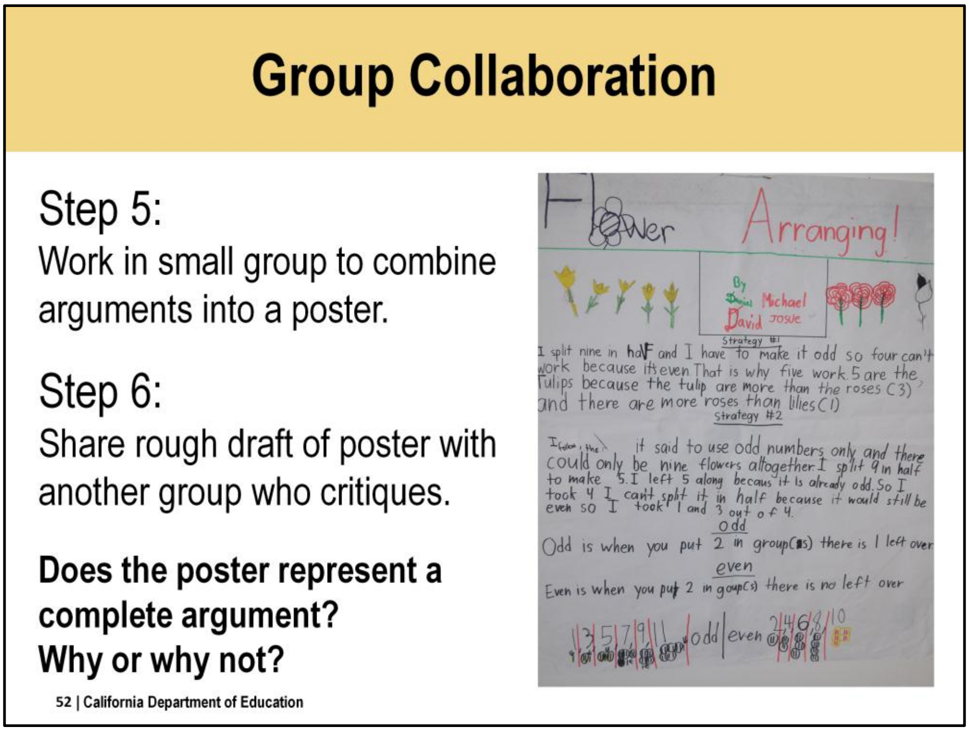
[refer to slide]




**Facilitator Notes:**

- Refer participants to "David, Josue, and Michael's Poster" **(Handout 3.4.4**)

**Talking Points:**

- The last two steps of the revision process include: [refer to steps on slide]
- Look at the poster that Daniel, Michael, David, and Josue developed, combining their strategies and representations.
- Consider the following questions: Does the poster represent a complete argument? Why or why not?

# Discussion

Reflect on the benefits of investing time and instruction to provide opportunities for students to revise and refine their arguments.

- **What do students learn from this process?**
- **How might this impact future learning?**

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**Facilitator Notes:**

[refer to slide for pair talk or small group discussion]

## Student-Generated Challenge

The class developed their own challenge:

What happens when adding odd and even numbers?

- Max, Spencer, and Haley worked on this challenge during their own time.

Refer to "Student-Generated Math Challenge" (Handout 3.4.4)

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**Talking Points:**

- Led by a student with special needs, the class developed their own challenge: What happens when adding odd and even numbers?
- Max, Spencer, and Haley worked on this challenge during their own time. [Refer to "Student-Generated Math Challenge" (**Handout 3.4.4**)
- This student-generated challenge fits the descriptors of Ball's reasoning community, stating how public knowledge (e.g., the definition of odd and even) becomes the foundation for new knowledge under construction (e.g., developing rules for adding odd and even numbers).

## Students Describing Viable Arguments

View this final odd/even video of three students describing viable arguments:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11852

*"Children must be shown how to cultivate a climate of debate, questioning and multiple interpretations. They must think about how to disagree with each other in ways that allow the other person to hear what is being said."*

Calkins, 2001

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**Notes to Facilitator:**

Movie length: 4 minutes. Prepare Internet video hyperlink:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11852

**Talking Points:**

- View this final odd/even video of three students describing viable arguments.

## 3.5 Summary: MP2

Compare the student responses to the Flower Arrangements task to the reasoning practices in *MP2. Reasoning Abstractly and Quantitatively* **(Handout 3.0.2)**.

Look for evidence of the practice descriptors in the student work samples just reviewed.

**What evidence from the student work on “Flower Arrangements” fit the descriptors for reasoning abstractly and quantitatively?**

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**Facilitator Notes:**

- Refer participants to the *MP2. Reasoning Abstractly and Quantitatively* section of the “Comparing Standards” document in their Participant Packet **(Handout 3.0.2).**
- Review slide.
- Discuss: What evidence from the student work on Flower Arrangements fit the above descriptors for reasoning abstractly and quantitatively?

# Summary, MP3

Refer to *MP3. Constructing Viable Arguments and Critiquing the Reasoning of Others* (Handout 3.0.2).

- Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions.
- Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.

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**Facilitator Notes:**

- Refer participants to the *MP3. Constructing Viable Arguments and Critiquing the Reasoning of Others* section of the "Comparing Standards" document in their Participant Packet **(Handout 3.0.2)**.
- As demonstrated in this unit, elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.

# Community of Reasoners

Students in the participating classroom made strides in building a strong community of reasoners.

**Mathematical Conversations**

-Active listeners

-Analyze what have heard

-Ask clarifying questions

-Build on other's ideas

-Challenge other's thinking

-Present a variety of strategies

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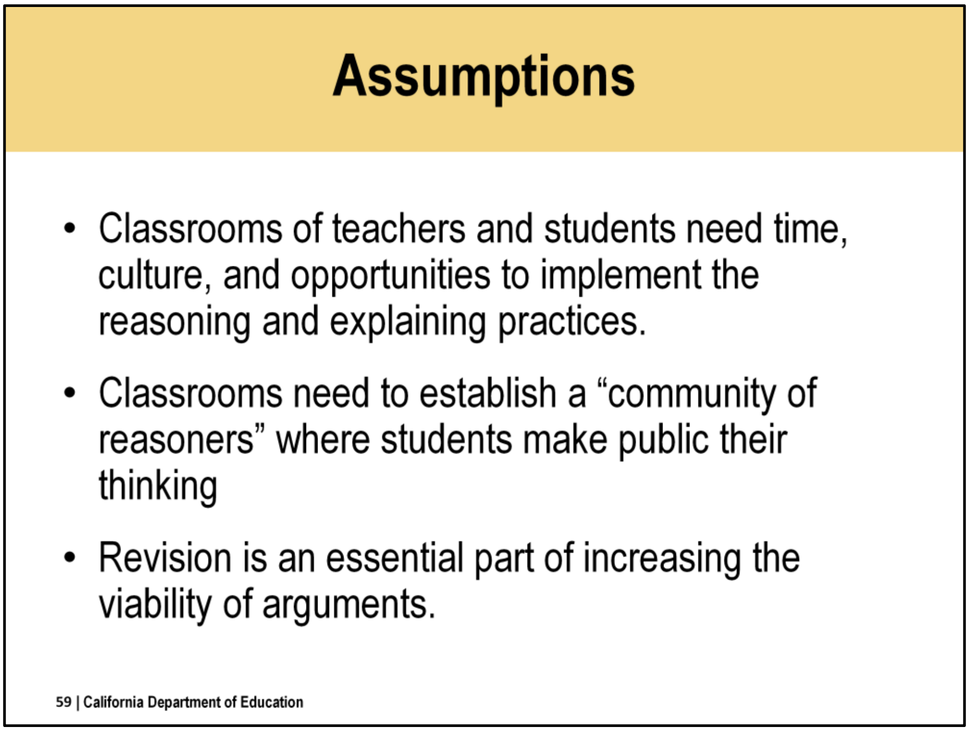
**Talking Points:**

- Developing a reasoning community is time-consuming and intensive, but after the community is established, the energy expended transitions to applying reasoning not only to mathematics, but also other areas.
- For example, within the participating class, the physical education teacher noticed a difference in the genuine respect that students had for each other. On a field trip to the State Capitol, students asked the Governor's aide if the Governor had a viable argument for vetoing a certain bill.
- The classroom teacher said that developing arguments using evidence and questioning gave her students a voice to advocate for themselves and others.




**Note to Facilitator:**

[Review slide]

## Claim

If all student become proficient with using the reasoning and explaining practices, they will improve their learning, understanding, and application of mathematics.

*"Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments."*

California's CCSS for Mathematics, 2012

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**Note to Facilitator:**

[Review slide]

# Reflection

In your Metacognitive Journal, reflect on what you learned in this unit to find evidence of the reasoning and explaining standards.

Write about how you might go about creating a community of reasoners and implement the practices into your classroom.

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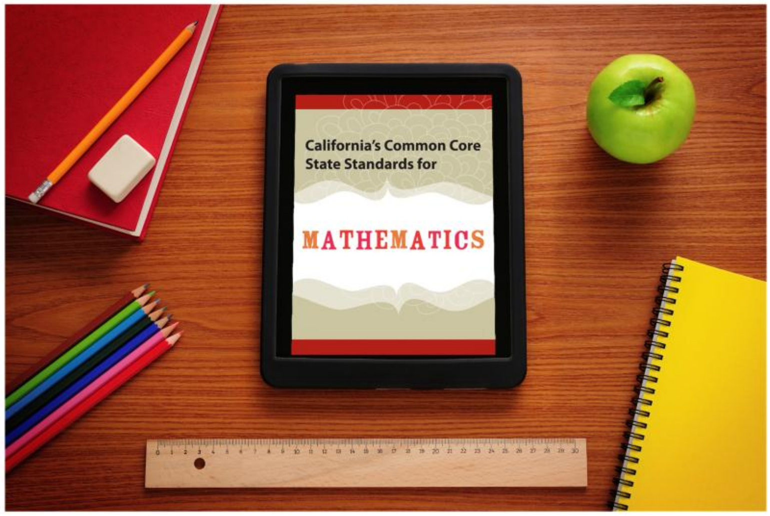
# California's Common Core State Standards for Mathematics




**(TRANSITION SLIDE)**