Kindergarten through Grade Twelve Standards for Mathematical Practice

Unit 3:

Reasoning and Explaining: MP2 and MP3

CALIFORNIA DEPARTMENT OF EDUCATION
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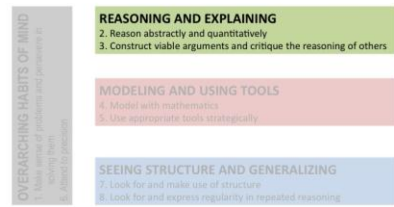
CCSS Mathematical Practices



Unit 3 Learning Objectives

- You will be able to describe why, to be successful in mathematics, all students need to reason and explain.
- You will be able to explain what it means for students to reason abstractly and quantitatively.
- You will be able to explain what it means for students to construct viable arguments and critique the reasoning of others.

Unit 3 Overview

Focus on how students learn to:

- Construct viable arguments
- Respond to the reasoning of others
- Increase the viability of their arguments

Unit 3 highlights 5th-grade students learning to use the reasoning and explaining practices.

Teachers of all grade levels can transfer concepts learned into their instructional practices.

Unit 3 Overview, Cont.

- Unpacking MP2 and MP3 and Introduction to 5th Grade Classroom
- Beginning to Reason: Definitions and Conjectures
- Explaining and Justifying
- Identifying Flaws in Reasoning
- Making Arguments More Viable
- Summary and Reflections

Unpacking MP2 and MP3

Read MP2 and MP3

- Highlight key words or phrases that seem particularly cogent to you or that puzzle or intrigue you
- Make a note of questions you have about particular parts of these two mathematical practices.
- Consider in particular how the two practices are related.

Small Group Discussion

- What key words or phrases did you highlighted and why they were important to you?
- What questions do you have about these two mathematical practices?
- How are the two standards related?
- What experiences do your students have representing solutions, sharing strategies, listening to other students, and questioning others?
- How can you use this information to build toward MP2 and MP3?

Noticing Students' Thinking

Throughout Unit 3 you will follow the same 5th-grade students as they learn to engage in viable arguments.

"If discussions about what constitutes a valid justification do not occur, students often rely on the superficial aspects of an argument, such as the use of formal mathematical symbols, over the mathematical reasoning that underlies the argument."

Healy & Hoyles, 2000

Introductions

The 5th-grade students featured in this unit participated in a public lesson study (coached by the California Mathematics Project) and focused on the reasoning and explaining practices.

- Students Represented: 44% English learners; 67% free and reduced lunch; 8% special needs

As you watch the video, consider how the students define and value viable arguments:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11837

What are the descriptors of viable arguments that Jeanette, Haley, and Leslie speak about?

How do they describe viable arguments as part of their learning process?

Connections Across Standards

Compare reasoning practices in the different standards:

- CCSS for Mathematics Standards
- ELA College and Career Readiness (CCR) Anchor Standards
- Partnership for 21st Century Skills

3.1 Beginning to Reason

- The CCSS for mathematical practices do not use the word "proof".
- The term "argument" is more general but has the same purpose of constructing a logical sequence of steps that justify and communicate a given conclusion from a set of agreed premises.

"Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments."

CCSS for Mathematics, page 1

Definitions and Conjectures

- When students begin to reason, they need to establish definitions (e.g., clear and logical descriptions of *what* they are reasoning about).
- They also need to make conjectures — informed guesses of what is true — they then set out to explain through reasoning.

Student Definitions of "Even"

- Complete "Odd and Even Survey"
- Analysis of student responses:
  - Does the student grade level impact logic and/or clarity?
  - What argument strategies (e.g., example, counter-examples, non-examples) are evident ?
  - What should be included in a definition to be considered complete and clear?

Reasoning

Reasoning of Inquiry

- Inquiry for discovering and exploring new ideas

Reasoning of Justification

- Justifying or proving mathematical claims.
  - Public knowledge – ideas, procedures, methods, terms established within a given community
  - Language – symbols, terms, notations, definitions, and representations and rules of logic and syntax

Developing a Community of Reasoners

Three domains of work (Ball, 2002):

- Select tasks and provide opportunities for reasoning
- Make knowledge public and scaffold the use of mathematical language and knowledge
  - Make mathematical records (through notebooks, public postings) to make the work public and available for public development
- Develop a classroom culture in which students
  - Develop serious interest in and respect for others' ideas
  - Attend and respond to, as well as use, others' solutions or proposals as a means of strengthening understanding

Meeting the Needs of Diverse Learners

"Research has demonstrated that vocabulary learning occurs most successfully through instructional environments that are language-rich, actively involve students in using language, require that students both understand both spoken and written words and also express that understanding orally and in writing, and require students to use in multiple ways over extended periods of time. To develop written and oral communication skills, students need to participate in negotiating meaning for mathematical situations and in mathematical practices that require output from students."

-CCSS Initiative, Application of the Standards for English Language Learners

If discourse improves learning for students, how can you support English learners in this community?

Conjectures

"Conjectures enable students to discover and construct 'new' mathematical knowledge by connecting what they are trying to learn to previous experience and knowledge."

Carlton, 1998

The process of making and testing mathematical conjectures about predictable outcomes can impact student learning by:

- Engaging students in learning as they become invested in learning if their conjectures are correct or not
- Stimulating students to think and reason
- Giving students a chance to confront their misconceptions or faulty ideas
- Helping students construct their knowledge in a way that leads to deeper understanding and reasoning abilities

Alt, 2012

Introducing Viable Arguments

Harold: What do you know about odd and even?

Leslie: Odd is something that doesn't, like you can't have it equal. Number 7 is 4 and 3, not 4 and 4, like 8.

Max: Odd numbers if you add it, you'd have to add two different numbers, not like the same; and if you divide it, you get 1 remainder.

Daniela: Odd numbers don't have partners, and even numbers do have patterns. Four, there are two fingers for each. Three has one left over. It doesn't have a partner, so it's not even.

The following video captures Daniela's pairing of fingers:
http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11843

Misconception: “50 is both odd and even”

Instructional conjecture: If students view a video of another student’s misconception of odd and even, they would be motivated to develop an argument to help this student clarify her understanding. In turn, the class would develop a common definition that would become part of the classroom’s public knowledge.

A video of Emily was filmed immediately after she successfully participated in a textbook lesson defining even numbers as ending in 0, 2, 4, 6, 8:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11844

Identify the cases that Emily shared.

What is her misconception?

How would you help her?

Forming Conjectures

Daniela’s explanation:

“We figured out that 50 is an even number. And how we found that out is … because we first wanted to prove that (grouping) in any way, meaning in threes, fours, fives, or tens … we wanted to prove that it is even. So we tried (dividing into groups of) three, and it worked. But we had two left over, and we figured out that that’s still even, because the two has a partner. So it was even in any way, even if you have leftovers, the leftovers have even blocks.”

Achieving Consensus

“Evidence shows that class discussion is important in students’ development of mathematical conceptions … instances of disagreement arise from diverse ideas generated by children.”

Wood, 1999

As the work progressed, the students developed two conjectures the class agreed to:

- If you put cubes into pairs, and there are no leftovers, then the number is even.
- If you put the cubes into pairs, and there is a leftover, then the number is odd.

Other Conjectures

- Even means the “same amount.” If you divide a number by any divisor, and it doesn’t have a leftover, then it’s even.
- If you divide a number by some divisor and there is no leftover, then the number is even. If you divide the same number by a different divisor, and there is a leftover, then that shows that the number is odd. For example:

15 ÷ 3 = 5 (even)
15 ÷ 2 = 7 R-1 (odd)

... so 15 is both odd and even …

Discuss

In small groups discuss:

- Allotting time for students to develop thinking around definitions, content, and argument
- Developing a community of reasoners that includes: English learners, students with special needs, and high-achieving students
- Curriculum topics that offer students opportunities for developing conjectures

3.2 Explaining and Justifying

“Higher order questions generally challenge the student to provide additional information and engage in deeper understanding and reflection, and ultimately promote greater conceptual development .”

Nathan & Kim, 2007

Taxonomy of Questions in Mathematical Discourse

When students make their thinking visible, they are organizing their thoughts, connecting to previous understandings, and recognizing and correcting gaps in their logic.

- Review the “Taxonomy of Questions in Mathematical Discourse” handout.

An awareness of language patterns enables the teacher and other students to access student thinking which helps to develop a public knowledge base. This awareness is especially helpful in supporting English learners and students with special needs.

Levels of Explanations

Level 1: Abdicating reasoning to an external authority

Level 2: Systematically working through multiple explanations

Level 3: Beginning to notice patterns or trends

Level 1: External authority

First level of explanations:
Deferring to an external authority

As you watch the video, think about how attributing thinking to someone else impacts student learning.

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11845

Level 1: Perceptual Proof

- Above are single examples demonstrating what an odd number is.
- In the video, students are satisfied with one case as sufficient evidence to validate their claim. This is referred to as a “perceptual proof.”

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11846

Level 2: Multiple Examples

Second level of explanations:
Learning to present an argument by presenting multiple examples or cases

- Demonstrates that a claim works in more than one instance.
- Students often get into an argument about how much evidence is enough, especially when they realize they cannot test every case.
- In the video, Brandon, Chris, and Nathan use multiple examples organized systematically:

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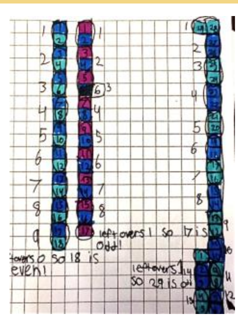
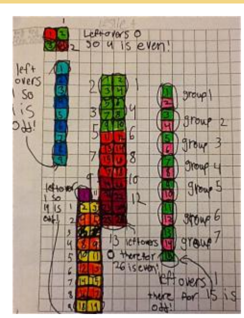
Level 2: Multiple Examples





Level 3: Noticing Patterns

Third level of explanations:
Multiple representations with emerging pattern(s)

Students begin to see patterns across multiple representations.

- Being able to identify and understand these patterns supports learners as they begin to abstract and generalize.
- Review Handout 3.2.2, "Level 3: Noticing Patterns Across Multiple Examples"

Level 3: Patterns Across Examples

Watch Brandon, Chris, and Nathan working across strategies:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11848

Discuss in small groups:

- The evidence students use in their explanation to argue for their claim.
- Does the argument include elements of an accurate generalization?
- How will this impact their understanding of this concept?
- How does the work with explanations presented in this unit intersect with the recommendations for English learners? Students with special needs? (Refer to Handout 3.2.3 "Engaging Diverse Learners")

3.3 Flaws in Reasoning

"Argument is central to thought and the construction of knowledge (e.g., Kuhn, 1992). The significance of argument to conceptual understanding in mathematics is related to the development of students' thinking and reasoning that occurs during the acts of challenge and justification."

Wood, 1999

"Students learn about mathematics by exploring their own and others' reasoning in problem-solving situations. Exploring the reasoning of self and others allows for flaws in thinking to be revealed and corrected. Opportunities for students to reveal their thinking and for their peers to evaluate and contribute to the improvement of student thinking can lead to stronger mathematical understanding. As students become more mathematically proficient and their reasoning skills increase they should be able to identify flaws in their own and others' thinking; thus prompting revision of thinking that leads to better problem solving."

Daro, 2012

Common Flaws

Several common flaws emerged as the 5th-grade students made their arguments part of the public conversation.

Competing definitions
- A number that can be divided evenly by any divisor without a remainder
- A number that is a multiple of two

Pronoun referents

"Odd numbers if you add it, you'd have to add two different numbers, not like the same; and if you divide it, you get one remainder."

Partial understanding or explanations
- "Odd means there is a number left out when you count by 1s, 3s, 5s, 7s, or 9s."
- "I think two decides, because two is always even."

Identifying Flaws through Discourse

Discussion: Refer to Handout 3.3.1

Community of Reasoners:
- Who is leading the conversation?
- Do you see a respect for others' mathematical ideas? What is the evidence?
- How often does the teacher intervene and for what purpose(s)?
- Where do you see evidence of students referring to, building on, and/or challenging other's thinking?

Arguments:
- What questions do the students pose?
- What reasoning does Chris use? Max?
- What flaws are students identifying?

Flaws:
- Does language cause confusion?
- Are there partial arguments?
- What is missing?

Language of Explanations

"(Teacher's work is to) establish a classroom culture permeated with serious interest in and respect for others' mathematical ideas. Deliberate attention is required for students to learn to attend and respond to as well as use, others' solutions or proposals as a means of strengthening their own understanding and the subsequent contributions they can make to the class' work."

Ball 2002

The language of explanations uses conjunctions (because, but, so, and, etc.).

Tristin: **'Cause** usually when you count by fours, it's even: 4, 8, 12, 16, 20. Those are all even. Then 3, 6, 9, 12, 15 is: odd, even, odd, even.

Because usually like 2, 4, 6, 8 is all even. It just keeps going in a pattern.

Daniela: Yep, well every time you count by twos, it's even. **But**, the rest of the numbers are switching: odd, even, odd, even.

Language of Justification

The language of justification uses an "If…. then… because…" structure. In many cases, "then" is implied.

Max: I think what it's trying to say is that **if** it's 3, 3, and 3, (**then**) those are all equal numbers, **but** 3 is odd.

Justifications often include modals (would, should, could, might have, must). This language may need some additional support.

Chris: **If** you count by twos, **(then)** you **would** just get an even number, and you'll never land on an odd.
But like Tristin said, "…it goes in a pattern: odd, even, odd, even. And, **if** one number was both, **(then)** it **would** break the pattern. I don't think a number can be both even and odd."

Flaws in Reasoning

A persistent argument among the students was that a number could be both odd and even. Watch three students explain the flaw in the others' reasoning:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11849

Watch as three groups of students in the classroom demonstrated misunderstandings when asked to define what a mathematical "rule" was:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11850

Teachers may assume that students understand that a mathematical rule is a statement that is always true. The definition exercise demonstrated that in the daily activities of students, rules often have exceptions.

3.4 Making Arguments More Viable

"*As students become more mathematically proficient and their reasoning skills increase they should be able to identify flaws in their own and others' thinking: thus prompting revision of thinking that leads to better problem solving*."

Daro, 2012

Public Definitions of Odd and Even

Before the revision work began, the class agreed to the following definitions of odd and even:

– An even number ends in 0, 2, 4, 6, or 8
– If you count by twos, you will land on an even number
– An even number can be divided by 2 and has no remainder
– An odd number ends with 1, 3, 5, 7, or 9
– When you put numbers into pairs, and there is a leftover, the number is odd

Revision Sequence

Step 1: Complete the MARS Flower Arrangements task
Step 2: Meet individually with the teacher, then revise work
Step 3: Share solution with a partner and revise work
Step 4: Share solution with whole class and revise
Step 5: Work with small group to combine arguments into a poster
Step 6: Share rough draft of poster with another group and revise
Step 7: Share completed poster at public research lesson

MARS Task: Flower Arrangements

Flower Arrangements: Discussion

Step 1: Complete the "Flower Arrangements" MARS task (Handout 3.4.1).

- Consider how an English learner, GATE, and special needs student might respond to the task.

Discuss:

- Constraints of the problem (e.g., there are more roses than lilies). Note: Students used "clue" as a synonym for "constraint".
- Questions you would ask students if they skipped any one of these constraints.

Flower Arrangements: Student Work Sample 1

Chris:

"I did addition and then I just guessed a random number for tulips; then I remembered that there is more tulips than roses and roses than lilies, so I went down by 2 number and then added 5 + 3 + 1 and it equaled 9. I was thinking of an even number being a problem but then in the direction it said, 'She does odd numbers'."

Flower Arrangements: Student Work Sample 2

Nathan:

"I figured it out with the clues. Tim's grandmother always uses an odd number, and there are more tulips than roses and more roses than lilies. She is using 9 flowers today which is an odd number [II II II II I]. 5 is odd [II II I], because there is one left out. 3 is odd, because there is one left out [II I]; and 1 is odd, because it has no partner."

Flower Arrangements: Student Work Sample 3

Spencer:

"I figured this out by looking at the odd numbers and seeing which one I could use without going too high. So I chose numbers 5, 3, 1, and they add up to 9. You can't get to an odd number by going by 2."

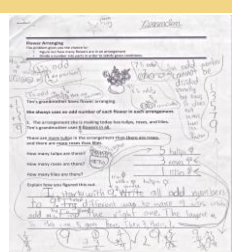
Quick Conference with Teacher

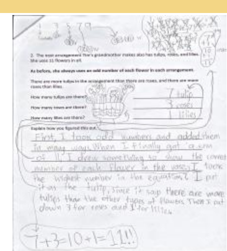
Step 2: Students individually meet with teacher, then revise work

Teacher prompts:

- How do you know that 5 is an odd number?
- How did you know that it was 5 and not 7?
- Can you show me another way?
- Why doesn't 4 + 3 + 2 work? It totals 9…
- What does 5 refer to? The 3? The 1?
- Could you use what you learned in the first problem to help with the second problem?

Brandon's Revisions





Does Brandon meet all the constraints of the problem?
How do the additions to his work impact his response? (arrows point to revisions)
Is anything missing?
What would you ask him to consider about his run-on equation: 7 + 3 = 10 + 1 = 11?

Partner Work

Step 3: Students share with a partner and revise

- Often, when a student is explaining a solution to someone else they discover gaps, mistakes, or incomplete responses.
- Refer to “Partners Sharing” (Handout 3.4.3) and look for points where Leslie and Stephanie correct themselves or refine their argument as they are talking.

Using Objects to Share Solutions

Step 4: Students share work with the whole class then revise

Explaining individual solutions allows for rehearsing an argument and opening it up to questioning.

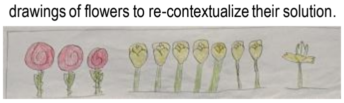
Public sharing also gives the students an opportunity to view multiple representations and strategies.

Using Objects: Finger Method, Tally Marks, and Drawings

Daniela uses her fingers as a tool. She begins by putting out nine fingers and then adjusts the numbers of fingers to meet the constraints of the problem.

Some students used tally marks to decontextualize the problem.

Other students decontextualized the problem using lists of odd numbers and cubes, and then used drawings of flowers to re-contextualize their solution.

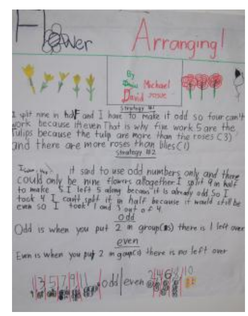


Group Collaboration

Step 5:
Work in small group to combine arguments into a poster.

Step 6:
Share rough draft of poster with another group who critiques.

Does the poster represent a complete argument? Why or why not?



Discussion

Reflect on the benefits of investing time and instruction to provide opportunities for students to revise and refine their arguments.

- **What do students learn from this process?**
- **How might this impact future learning?**

Student-Generated Challenge

The class developed their own challenge:
What happens when adding odd and even numbers?

- Max, Spencer, and Haley worked on this challenge during their own time.

Refer to “Student-Generated Math Challenge” (Handout 3.4.4)

Students Describing Viable Arguments

View this final odd/even video of three students describing viable arguments:

http://myboe.org/portal/default/Content/Viewer/Content?action=2&scId=306591&sciId=11852

"Children must be shown how to cultivate a climate of debate, questioning and multiple interpretations. They must think about how to disagree with each other in ways that allow the other person to hear what is being said."

Calkins, 2001

3.5 Summary: MP2

Compare the student responses to the Flower Arrangements task to the reasoning practices in *MP2. Reasoning Abstractly and Quantitatively* **(Handout 3.0.2)**.

Look for evidence of the practice descriptors in the student work samples just reviewed.

What evidence from the student work on "Flower Arrangements" fit the descriptors for reasoning abstractly and quantitatively?

Summary, MP3

Refer to *MP3. Constructing Viable Arguments and Critiquing the Reasoning of Others* (Handout 3.0.2).

- Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions.
- Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.

Community of Reasoners

Students in the participating classroom made strides in building a strong community of reasoners.

Mathematical Conversations

-Active listeners

-Analyze what have heard

-Ask clarifying questions

-Build on other's ideas

-Challenge other's thinking

-Present a variety of strategies

Assumptions

- Classrooms of teachers and students need time, culture, and opportunities to implement the reasoning and explaining practices.
- Classrooms need to establish a "community of reasoners" where students make public their thinking
- Revision is an essential part of increasing the viability of arguments.

Claim

If all student become proficient with using the reasoning and explaining practices, they will improve their learning, understanding, and application of mathematics.

"Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments."

California's CCSS for Mathematics, 2012

Reflection

In your Metacognitive Journal, reflect on what you learned in this unit to find evidence of the reasoning and explaining standards.

Write about how you might go about creating a community of reasoners and implement the practices into your classroom.

California's Common Core State Standards for Mathematics